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On the Linearized Relativistic Boltzmann Equation. II. Existence of Hydrodynamics

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Solutions are analyzed of the linearized relativistic Boltzmann equation for initial data from $L^2(r, p)$ in long-time and/or small-mean-free-path limits. In both limits solutions of this equation converge to approximate ones constructed with solutions of the set of differential equations called the equations of relativistic hydrodynamics.

KEY WORDS: Kinetic theory; relativistic Boltzmann equation; relativistic hydrodynamics.

1. INTRODUCTION

One of the major problems in relativistic statistical mechanics is to provide a consistent derivation of the equations of hydrodynamics. In spite of the effort of many authors,[1-6] this problem is still open. There is even deep confusion in this subject due to various formulations of basic postulates and consequently various sets of both hydrodynamic variables and equations. Among the best-known versions of relativistic hydrodynamics are the works of Eckart,[1] Landau and Lifschitz,[2] Israel,[3] Israel and Stewart,[4] Liu *et al.*,[5] and van Kampen.[6] What is even worse is that there is still no definite experimental evidence in favor of any of these theories. On the other hand, in many physical situations, such as the description of the quark plasma found in heavy ion collisions[7] as well as in many models used in astrophysics,[8] one needs certain hydrodynamic equations.

Here I want to avoid the arbitrariness connected with existing derivations of relativistic hydrodynamics and prefer to assume that on the microscopic level the system can be accurately described by the relativistic

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Boltzmann equation.[9,10] My aim is to prove that in certain well-defined limits a solution of this equation can be approximated by a solution of a properly chosen set of differential equations I then call the equations of relativistic hydrodynamics.

Similar to nonrelativistic theory,[11 15] the main ingredients of such an approach are certain integral estimates on the collision part of the linearized Boltzmann operator. These were done in ref. 16 (which hereafter will be referred to as I). Here I confine myself to a subclass of scattering cross sections analyzed in I, namely to cross sections corresponding to so-called cutoff hard interactions. For such a class of cross sections one is able to perform fairly detailed analyses of spectral properties of the linearized relativistic Boltzmann operator and find then an exact form of solution to this equation in the long-time and/or small-mean-free-path limits. Next I postulate a closed set of differential equations for moments of five conserved quantities, i.e., density of particles and density of momentum and energy, and prove the existence of global solutions to these equations in Sobolev spaces $H^l(r)$. With the help of these five moments I construct an approximation to the solution to the linearized Boltzmann equation and show that in both of the above-mentioned limits such an approximation agrees with the actual solution of the Boltzmann equation with corresponding initial data. I call this set the equations of relativistic hydrodynamics and in fact these are the same equations one can derive from the Boltzmann equation via the Chapman–Enskog expansion. I consider here a weaker form of the meaning of hydrodynamics as a reduced description of a system, governed by the Boltzmann equation, valid for the long-time and/or small-mean-free-path limits. However, if one assumes that the connection between the long-time asymptotics of the Boltzmann equation and the hydrodynamic description of the system holds in the relativistic theory as well as it holds in the nonrelativistic one, then the present approach leads at least to the correct asymptotic form of the equations of relativistic hydrodynamics. An outline of the results was given in ref. 24.

2. THE LINEARIZED RELATIVISTIC BOLTZMANN EQUATION

Let us consider a one-component classical relativistic gas of particles with rest mass $m \neq 0$ in flat space-time and in the absence of external forces. It is convenient to introduce dimensionless variables x^μ and p^μ related to the usual dimensional four-vectors of position y^μ and momentum q^μ by the relations $x^\mu = y^\mu/c\eta$ and $p^\mu = q^\mu c/kT$. The dimensionless mass is $M = mc^2/kT$. It is convenient to interpret T as the temperature of the corresponding global equilibrium state. η is a time scale to be specified.

Formulas are written in covariant manner, using an arbitrary chosen frame of reference. In this frame one decomposes x^μ and p^μ as follows:

$$x^\mu = (t, \mathbf{r}), \qquad p^\mu = (p_0, \mathbf{p})$$

where as usual $p_0 = (M^2 + \mathbf{p}^2)^{1/2}$.

The evolution of the one-particle distribution function $F(\mathbf{r}, \mathbf{p}, t)$ is governed by the relativistic Boltzmann equation:

$$\frac{\partial}{\partial t} F + \frac{\mathbf{p}}{p_0} \frac{\partial F}{\partial \mathbf{r}} = \int d_3 p_1 \, d\Omega g \frac{s^{1/2}}{2 p_0 p_{10}} \sigma(g, \Theta)[F'F_1' - FF_1] \tag{2.1}$$

where

$$s^{1/2} = |p_1 + p|$$

$$2g = |p_1 - p|$$

$$\cos \Theta = 1 - 2(p_\mu - p_{1\mu})(p^\mu - p'^\mu)(4M^2 - s)^{-1}$$

$$d\Omega = \sin \Theta \, d\Theta \, d\phi$$

$\sigma(g, \Theta)$ is the differential scattering cross section. All the above quantities refer to the center-of-mass frame; $F = F(\mathbf{r}, \mathbf{p}, t)$, $F_1 = F(\mathbf{r}, \mathbf{p}_1, t)$, $F' = F(\mathbf{r}, \mathbf{p}', t)$, and $F_1' = F(\mathbf{r}, \mathbf{p}_1', t)$.

For a system close to a global equilibrium its distribution function can be written as

$$F(\mathbf{r}, \mathbf{p}, t) = f_0(\mathbf{p}) + [f_0(\mathbf{p})]^{1/2} f(\mathbf{r}, \mathbf{p}, t) \tag{2.2}$$

where the relativistic equilibrium distribution function $f_0(p)$ has the form[17]

$$f_0 = \frac{n}{4\pi M^2 K_2(M)} \exp(-u^\mu p_\mu) \tag{2.3}$$

The u^μ has the meaning of dimensionless hydrodynamic four-velocity, and $K_2(M)$ is the Bessel function of the second kind of index 2.[18]

Substituting a function F in the form given by Eq. (2.2) into Eq. (2.1) and disregarding terms quadratic in f, we obtain the linearized relativistic Boltzmann equation (LRBE):

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{p_0} \nabla f &= \frac{f_0^{1/2}}{2p_0} \int d_3 p_1 \, d\Omega \frac{g s^{1/2}}{p_{10}} \sigma(g, \Theta) f_{10} \\ &\quad \times \left[\frac{f_1'}{(f_0)^{1/2}} + \frac{f'}{(f_0)^{1/2}} - \frac{f_1}{(f_0)^{1/2}} - \frac{f}{(f_0)^{1/2}} \right] \\ &\equiv \frac{1}{\varepsilon} L[f] \end{aligned} \tag{2.4}$$

In the following I consider Eq. (2.4) in several functional spaces and now introduce these spaces and define norms needed later.

Let $L^p(\circ)$ ($\circ = \mathbf{r}, \mathbf{p}, \mathbf{k} \in \mathbf{R}^3$) denote the space of functions whose pth ($1 \leqslant p \leqslant \infty$) power is integrable with respect to the Lebesgue measure in $\mathbf{R}^3$. The usual norm in these space is denoted by $\|f(\circ)\|_{L^p(\circ)}$. $H^l(\mathbf{r})$ denotes the Sobolev subspace of $L^2(\mathbf{r})$ functions the derivatives of which up to and including order l belong to $L^2(\mathbf{r})$. $H^l(\mathbf{k})$ means the Fourier transform of $H^l(\mathbf{r})$ with a norm

$$\|f(\mathbf{k})\|_{H^l(\mathbf{k})} = \|(1+k^2)^{1/2} f(\mathbf{k})\|_{L^2(\mathbf{k})}$$
$$\equiv \|f(\mathbf{r})\|_{H^l(\mathbf{r})} \tag{2.5}$$

In this and following denote the Fourier transform of $f(\mathbf{r})$ by its argument, i.e.,

$$f(\mathbf{k}) = (2\pi)^{-3/2} \int \exp(-i\mathbf{kr})\, f(\mathbf{r})\, d_3 r$$

Call H_0 the $L^2(\mathbf{r}, \mathbf{p})$ with a norm

$$\|f\|_{L^2(\mathbf{r},\mathbf{p})} = \left(\int |f(\mathbf{r}, \mathbf{p})|^2\, d_3 p\, d_3 r \right)^{1/2} \tag{2.6}$$

Now define a partial Fourier transform in $\mathbf{r}$ of $f \in H_0$ as

$$f(\mathbf{k}, \mathbf{p}) = (2\pi)^{-3/2} \int d_3 r \exp(-i\mathbf{kr})\, f(\mathbf{r}, \mathbf{p}) \tag{2.7}$$

Denote $\hat{H}_0 = \{f(\mathbf{k}, \mathbf{p}) : f(\mathbf{r}, \mathbf{p}) \in H_0\}$ and define

$$\|f(\mathbf{k}, \mathbf{p})\|_{\hat{H}_0} = \int d_3 k\, d_3 p\, |f(\mathbf{k}, \mathbf{p})|^2 = \|f\|_{L^2(\mathbf{r},\mathbf{p})} \tag{2.8}$$

For all $l \geqslant 0$, define H_l as the Hilbert subspace of $L^2(\mathbf{r}, \mathbf{p})$ consisting of all $H^l(\mathbf{r})$-valued $L^2(\mathbf{p})$ functions with a norm

$$\|f(\mathbf{r}, \mathbf{p})\|_l = \left[\int \|f(\circ, \mathbf{p})\|^2_{H^l(\mathbf{r})}\, d_3 r \right]^{1/2}$$
$$\equiv \left\{ \iint [(1+k^2)^{1/2}\, |f(\mathbf{k}, \mathbf{p})|]^2\, d_3 k\, d_3 p \right\}^{1/2}$$
$$\equiv \|f(\mathbf{k}, \mathbf{p})\|_l \tag{2.9}$$

For a scattering cross section $\sigma(g, \Theta)$, assume that there exist constants α, β, $\tilde{\beta}$, γ, $\tilde{\gamma}$, B, B', $\tilde{B}$, and C_0 such that

$$\gamma > -2, \qquad 0 < \alpha < \min(4, 4+\gamma), \qquad 0 < \beta \leqslant \gamma + 2$$
$$\tilde{\gamma} > -2, \qquad 0 \leqslant \tilde{\beta} \leqslant \tilde{\gamma} + 2$$

and the following are fulfilled:

(i) $$\sigma(g, \Theta) \leqslant (Bg^{\beta} + B'g^{-\alpha}) \sin^{\gamma} \Theta \tag{2.10a}$$

(ii) $$\sigma(g, \Theta) \geqslant \tilde{B} \frac{g^{\tilde{\beta}+1}}{c_0 + g} \sin^{\tilde{\gamma}} \Theta \tag{2.10b}$$

As shown in I, the assumption (2.10a) makes it possible to express the relativistic collision operator L on the rhs of Eq. (2.4) in the following form:

$$\frac{1}{\varepsilon} L = \frac{1}{\varepsilon} [-v(\mathbf{p}) + K_2 - K_1]$$

where

$$K_i[f(\mathbf{r}, \mathbf{p}, t)] = \int d_3 p_1 k_i(\mathbf{p}, \mathbf{p}_1) f(\mathbf{r}, \mathbf{p}, t), \qquad i = 1, 2$$

$$k_1(\mathbf{p}, \mathbf{p}_1) = \frac{g s^{1/2}}{p_0 p_{10}} \exp\left[-\frac{(\tau + \tau_1)}{2}\right] \int_0^{\pi} d\Theta \sin\Theta \sigma(g, \Theta)$$

$$k_2(\mathbf{p}, \mathbf{p}_1) = \frac{1}{4} \frac{s^{3/2}}{g p_0 p_{10}} \int_0^{\infty} dx \exp\left[-\frac{(1+x^2)^{1/2}(\tau+\tau_1)}{2}\right]$$
$$\times \sigma\left[\frac{g}{\sin(\psi/2)}, \psi\right] \frac{1 + (1+x^2)^{1/2}}{(1+x^2)^{1/2}} I_0\left[\frac{|\mathbf{p} \wedge \mathbf{p}_1|}{2g} x\right]$$

$$v(p) = \int d_3 p_1 k_1(\mathbf{p}, \mathbf{p}_1) \exp[(\tau - \tau_1)/2]$$

and

$$\tau = u^{\mu} p_{\mu}, \qquad \tau_1 = u^{\mu} p_{1\mu}$$
$$\sin(\psi/2) = 2^{1/2} g [g^2 - M^2 + (g^2 + M^2)(1 + x^2)^{1/2}]^{-1/2}$$

$\mathbf{p} \wedge \mathbf{p}_1$ is a vector product of $\mathbf{p}$ and $\mathbf{p}_1$ calculated in the rest frame of the gas [in this frame $u^{\mu} = (1, 0, 0, 0)$]; the explicit expression for $|\mathbf{p} \wedge \mathbf{p}_1|$ has the form

$$|\mathbf{p} \wedge \mathbf{p}_1| = [4g(\tau\tau_1 - g^2 - M^2) - M^2(\tau - \tau_1)^2]^{1/2}$$

I have introduced explicitly the small parameter

$$\varepsilon = 2M^2 K_2(M) n^{-1} = (m^2/kT)\, K_2(m/kT) n^{-1}$$

One sees that ε is related to the mean free path in the corresponding equilibrium state and $\varepsilon \to 0$ as $n \to \infty$.

It was shown in I that the operator K is a compact operator in $L^2(\mathbf{p})$. The condition (2.10b) ensures that the function $\nu(\mathbf{p})$ is a real function and there exist two positive constants ν_0 and $\tilde{\nu}_0$ such that

$$0 < \nu_0 \leqslant \nu(\mathbf{p}) \leqslant \tilde{\nu}_0 (p_0)^{\lambda}, \qquad \lambda > 0$$

Thus, Eq. (2.10b) defines the so-called hard interactions. The coefficient λ is related to the constant $\tilde{\beta}$; the details of the proof of these statements are given in I.

3. GENERAL PROPERTIES OF THE RELATIVISTIC BOLTZMANN OPERATOR

It was shown in the previous section that for $\sigma(g, \Theta)$ fulfilling conditions (2.10) the LRBE can be written as

$$\partial_t f + \frac{\mathbf{p}}{p_0} \nabla f = \frac{1}{\varepsilon} L[f] \tag{3.1}$$

I describe now general properties of the operator L that will be needed later.

(i) L is a closed, unbounded, self-adjoint operator on $L^2(\mathbf{p})$ with a domain

$$D(L) = D(\nu(\mathbf{p})) := \{ f \in L^2(\mathbf{p}) \colon \nu(\mathbf{p}) f(\mathbf{p}) \in L^2(\mathbf{p}) \} \tag{3.2}$$

(ii) For all $f \in D(L)$, L is nonpositive operator, i.e.,

$$(f, Lf) \leqslant 0 \tag{3.3}$$

$$Lf = 0 \qquad \text{iff} \quad f \in N_0 := \{ f_0^{1/2}, p_x f_0^{1/2}, p_y f_0^{1/2}, p_z f_0^{1/2}, p_0 f_0^{1/2} \} \tag{3.4}$$

Denote the spectrum of the operator L as $\sigma(L)$. Then:

(iii) $\sigma(L)$ consists of discrete and continuous parts. The latter is identical with a set of values assumed by $-\nu(\mathbf{p})$ for all possible values of $\mathbf{p} \in \mathbf{R}^3$, i.e., with the interval $]-\infty, -\nu_0]$. By a suitable choice of the time scale η we can set the $\nu_0 = 1$. The discrete part of $\sigma(L)$ consists of eigenvalues of finite multiplicity which can accumulate only at the boundary of the continuous spectrum. Denote

$$\nu_1 := \sup\{\sigma(L) \cap [-\infty, -\beta]; \beta > 0\}$$

Because zero is not an accumulation point of the spectrum, it follows that $\nu_1 > 0$.

For $f \in H_l(\mathbf{r}, \mathbf{p})$, $l \geq 0$, we can perform a Fourier transform in $\mathbf{r}$ of Eq. (3.1) and we obtain

$$\partial_t f(\mathbf{k}, \mathbf{p}, t) = \frac{1}{\varepsilon} B_{k\varepsilon}[f(\mathbf{k}, \mathbf{p}, t)] \tag{3.5}$$

where

$$B_{k\varepsilon} = -\nu(\mathbf{p}) + K - i\mathbf{k}\varepsilon \frac{\mathbf{p}}{p_0} \tag{3.6}$$

In the following we will be mainly concerned with Eq. (3.5). I list below some important properties of the operator $B_{k\varepsilon}$ that will be used frequently. As $|\mathbf{p}/p_0| \leq 1$, it follows that for every $\mathbf{k} \in \mathbf{R}^3$ and $\varepsilon \in [0, 1]$, $D(B_{k\varepsilon}) = D(L) = D(\nu(\mathbf{p}))$. Considering $B_{k\varepsilon}$ as an operator in $L^2(\mathbf{k}, \mathbf{p})$, we have:

(i) $B_{k\varepsilon}$ is closed, unbounded, and not self-adjoint.

(ii) $\mathrm{Re}(f, B_{k\varepsilon} f) \leq 0$, i.e., $\sigma(B_{k\varepsilon})$ lies in the left half-plane.

(iii) $\sigma(B_{k\varepsilon})$ is symmetric with respect to the real axis; if $\phi(\mathbf{k}, \mathbf{p})$ is an eigenfunction belonging to the eigenvalue λ, then $\phi(-\mathbf{k}, \mathbf{p})^*$ is an eigenfunction belonging to λ^*.

An important property of the spectrum $\sigma(B_{k\varepsilon})$ is described by the following:

Proposition 3.1. For all $\mathbf{k} \in \mathbf{R}^3$ and $\varepsilon \in [0, 1]$, $\sigma(B_{k\varepsilon})$ consists of a continuous part $\Gamma := \{-[ik\varepsilon\mathbf{p}/p_0 + \nu(\mathbf{p})]; \mathbf{p} \in \mathbf{R}^3\}$ and a discrete part of the isolated eigenvalues with finite multiplicity. These eigenvalues can accumulate only at the boundary of Γ.

Proof. We consider the operator $B_{k\varepsilon}$ as a perturbation of the operator $A_{k\varepsilon} = -\nu(\mathbf{p}) - i\mathbf{k}\varepsilon\mathbf{p}/p_0$ with a compact operator K. The spectrum of $A_{k\varepsilon}$ is purely continuous and $\sigma(A_{k\varepsilon}) = \Gamma$. The theorem of Schechter[19] ensures that the essential spectrum of $B_{k\varepsilon}$, i.e., $\sigma_{\mathrm{ess}}(B_{k\varepsilon})$ is equal to $\sigma_{\mathrm{ess}}(A_{k\varepsilon})$, this last being identical with $\sigma_{\mathrm{cont}}(A_{k\varepsilon})$, i.e., with Γ. As the operators $A_{k\varepsilon}$ and $B_{k\varepsilon}$ are closed, a general theorem states (ref. 20, p. 238) that the semi-Fredholm set of $B_{k\varepsilon} = A_{k\varepsilon} + K$ is equal to the semi-Fredholm set of the operator $A_{k\varepsilon}$. The semi-Fredholm set of the operator $A_{k\varepsilon}$ is characterized as follows. Let $S(k\varepsilon) = \{z \in \mathbf{C}\colon\ z = -\nu(\mathbf{p}) - i\mathbf{k}\varepsilon\mathbf{p}/p_0$ for $\mathbf{p} \in \mathbf{R}^3\}$. Then $S_F(\mathbf{k}\varepsilon) = \mathbf{C}/S(\mathbf{k}\varepsilon)$ is the semi-Fredholm for $A_{k\varepsilon}$ and also for $B_{k\varepsilon}$. The $S(\mathbf{k}\varepsilon)$ is a closed set, and thus $S_F(k\varepsilon)$ can be written in general as $S_F(\mathbf{k}\varepsilon) = S_F^1(\mathbf{k}\varepsilon) + S_F^2(\mathbf{k}\varepsilon)$, where both the $S_F^i(\mathbf{k}\varepsilon)$ are connected and $S_F^1(\mathbf{k}\varepsilon)$ contains the half-space $\{z\colon \operatorname{Re} z \geqslant 0\}$. As $B_{k\varepsilon}$ is closed, the nullity and deficiency of $B_{k\varepsilon} - z$ are both constants on the connected components of $S_F(\mathbf{k}\varepsilon)$, except for isolated values of z. It was shown in I that $B_{k\varepsilon}$ generates a contraction semigroup on $L^2(\mathbf{k}, \mathbf{p})$ and it follows then that both the nullity and deficiency of $B_{k\varepsilon} - z$ are zero for $\{z\colon \operatorname{Re} z > 0\}$. This means of course that they are zero for $z \in S_F^1(\mathbf{k}\varepsilon)$ except for some isolated values of z. We see, then, that, except for these values of z, $S_F^1(\mathbf{k}\varepsilon)$ is contained in the resolvent set of the operator $B_{k\varepsilon}$. This shows that these exceptional values of $z \in S_F^1(\mathbf{k}\varepsilon)$ can be only eigenvalues of finite multiplicity. ∎

4. RESULTS FROM THE PERTURBATION THEORY

Information concerning the local behavior of the eigenvalues and eigenfunctions of the operator $B_{k\varepsilon}$ for small values of $|\mathbf{k}\varepsilon|$ follows from the theory of analytical perturbations.[20,21] In order to apply this theory to the operator $B_{k\varepsilon}$, consider a family of operators B_γ defined as

$$B_\gamma = L + \gamma V \tag{4.1}$$

where $\gamma \in \mathbf{C}$ and I have introduced a bounded in $L^2(\mathbf{p})$ operator V,

$$V = \frac{\mathbf{kp}}{|k|\, p_0} \tag{4.2}$$

It is easy to check that for $\gamma = -i\,|\mathbf{k}\varepsilon|$ the operator B_γ is equal to the $B_{k\varepsilon}$. For all γ, $D(B_\gamma) = D(L)$ and one sees that for every $\phi \in D(L)$, $B_\gamma[\phi]$ is a vector analytic function of γ. Thus, the B_γ form an analytic family of the type A in the sense of Kato.[20,21] Moreover, for γ real, B_γ are self-adjoint. We can then apply the Kato–Rellich theorem (ref. 21, p. 22).

Theorem 4.1. (Kato–Rellich) Let B_γ be an analytic family in the sense of Kato for γ near 0 that is self-adjoint for real γ. Let λ_0 be a discrete eigenvalue of multiplicity m. Then, there are m not necessarily distinct single-valued functions analytic near $\gamma = 0$; $\lambda^1(\gamma) \cdots \lambda^m(\gamma)$, with $\lambda^i(0) = \lambda_0$, so that $\lambda^1(\gamma) \cdots \lambda^m(\gamma)$ are eigenvalues of B_γ near λ_0 for $|\gamma|$ small enough.

Theorem 4.1 ensures the existence of eigenvalues of the operator $B_{k\varepsilon}$ for sufficiently small $|k\varepsilon|$. We see that eigenvalues of the operator L cannot suddenly disappear if the perturbation $i\varepsilon\mathbf{kp}/p_0$ is turned on. These perturbed eigenvalues can be calculated by the standard perturbation expansion and this is done in Appendix A. These results make it possible to prove the following refinement of this general theorem:

Theorem 4.2. Let $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ denote the eigenvalues and corresponding eigenfunctions of the operator $B_{k\varepsilon}$. There exists $\delta>0$ such that for $|\mathbf{k}\varepsilon|\leqslant\delta$ the following are fulfilled:

(i) The eigenvalues $\{\lambda_j(k\varepsilon)\}_{j=1}^{5}$ are semisimple and the corresponding spectral projectors $P_j(\mathbf{k}\varepsilon)$ can be represented as

$$P_j(\mathbf{k}\varepsilon)f=\sum_{l=1}^{m_j}(e_l(-\mathbf{k}\varepsilon),f)_{L^2(\mathbf{p})}\,e_l(\mathbf{k}\,\varepsilon) \tag{4.3}$$

where m_j is the multiplicity of the eigenvalue $\lambda_j(k\varepsilon)$,

$$(e_l(-\mathbf{k}\varepsilon),e_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})}=\delta_{lj} \tag{4.4}$$

$\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ have expansions

$$\lambda_j(k\varepsilon)=\sum_{n=0}^{\infty}\lambda_{jn}[u^\mu](ik\varepsilon)^n \tag{4.5}$$

$$e_j(\mathbf{k}\varepsilon)=\sum_{n=0}^{\infty}e_{jn}[\mathbf{p},\mathbf{k}/k,u^\mu](ik\varepsilon)^n \tag{4.6}$$

λ_{jn} are functions of hydrodynamic velocity u^μ alone, $\lambda_{j2}>0$, and e_{jn} are functions of $\mathbf{p}$, $\mathbf{k}/k$, and u^μ.

(ii) $\sigma(B_{k\varepsilon})\cap[-(2/3)\nu_1,\ -(1/3)\nu_1]=\varnothing$.

Proof. The point $k=0$ is an exceptional point of the operator $B_{k\varepsilon}$, but at $k=0$, $B_0\equiv L$ is a symmetric operator, and thus a general theorem[20] on the perturbation of symmetric operators ensures that the eigennilpotents $D_j(k\varepsilon)$ vanish identically because they vanish for $k=0$. This means that $\lambda_j(k\varepsilon)$ are semisimple. For $k\neq0$ the fivefold degenerate zero eigenvalue of L splits into several groups λ_i each of multiplicity m_i such that $\sum m_i=5$. The same is of course true for the adjoint operator $\bar{B}_{k\varepsilon}=B_{-k\varepsilon}\equiv L+i\mathbf{kp}\varepsilon/p_0$. If $\phi_j(\mathbf{k}\varepsilon)$ is an eigenfunction belonging to the eigenvalue $\lambda_j(k\varepsilon)$ of the operator $B_{k\varepsilon}$ then $\phi_j(-\mathbf{k}\varepsilon)$ is an eigenfunction belonging to the eigenvalue $\lambda_j(k\varepsilon)^*$ of the adjoint operator $\bar{B}_{k\varepsilon}$. We can write, then,

$$\begin{aligned}(\phi_n(-\mathbf{k}\varepsilon),B_{k\varepsilon}\phi_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})}&=\lambda_j(k\varepsilon)(\phi_n(-\mathbf{k}\varepsilon),\phi_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})}\\&=\lambda_n(k\varepsilon)(\phi_n(-\mathbf{k}\varepsilon),\phi_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})}\end{aligned}$$

and we see that for $\lambda_j(k\varepsilon) \neq \lambda_n(k\varepsilon)$

$$(\phi_n(-\mathbf{k}\varepsilon), \phi_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})} = 0$$

There are biorthogonality relations for eigenfunctions belonging to different groups of eigenvalues. As the $\lambda_j(k\varepsilon)$ are semisimple, we can choose in the subspace belonging to each $P_j(\mathbf{k}\varepsilon)$ eigenfunctions in such a way that we have also

$$(\phi_n(-\mathbf{k}\varepsilon), \phi_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})} = \delta_{nj}, \qquad n, j = 1,..., 5$$

Note that for purely imaginary k these relations are equivalent to the statement that eigenfunctions of the symmetric operator can be chosen as an orthonormal basis. For k real, as it is in our case, such relations are true only for sufficiently small $k\varepsilon$ when we can apply the Kato–Rellich theorem.

With the help of such a basis we can express each of the $P_j(\mathbf{k}\varepsilon)$ as

$$P_j(\mathbf{k}\varepsilon) f = \sum_{l=1}^{m_l} (e_l(-\mathbf{k}\varepsilon), f)_{L^2(\mathbf{p})}\, e_l(\mathbf{k}\varepsilon)$$

Theorem 4.1 ensures that there exists $\delta > 0$ such that for $k\varepsilon < \delta$ $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ are analytic functions of $k\varepsilon$ and we can expand them into series in this parameter; coefficients of these expansions are calculated in Appendix A.

The zero eigenvalue of the operator L is separated from the rest of the spectrum by a gap with width v_1. We see that by taking the value of δ sufficiently small we can confine the first five eigenvalues of the operator $B_{k\varepsilon}$ to the circle in the complex plane with radii less then $(1/3)v_1$ and the rest of the spectrum to the half-space $\mathrm{Re}(\lambda) < -(2/3)v_1$. ∎

All of our considerations have been done in an arbitrarily chosen frame of reference and this choice set a definite value of the hydrodynamic four-velocity u^μ. In the following I suppress the dependence of λ_j and e_j on u^μ so long as it leads to no confusion.

5. GLOBAL BEHAVIOR OF EIGENFUNCTIONS

The information concerning the local behavior of eigenfunctions of the operator $B_{k\varepsilon}$ for small values of $k\varepsilon$ is not sufficient to show boundedness of the resolvent $(z - B_{k\varepsilon})^{-1}$ for z from the strip $\{z: -\beta \leqslant \mathrm{Re}\, z \leqslant 0\}$. This we will need if we want to obtain an explicit form of the solution to Eq. (3.1).

The aim of this section is to prove that for large $k\varepsilon$ the spectrum of the operator $B_{k\varepsilon}$ is bounded away from the imaginary axis. In fact, we will prove even more, namely, that for $k\varepsilon \to \infty$ the discrete part of the spectrum $\sigma(B_{k\varepsilon})$ moves toward continuum (see ref. 15 for nonrelativistic results).

We first consider the eigenvalue problem for the operator $B_{k\varepsilon}$,

$$\left[-\nu(\mathbf{p}) - \frac{i\mathbf{k}\varepsilon\mathbf{p}}{p_0} + K\right]\phi_j = \lambda(k\varepsilon)\phi_j \tag{5.1}$$

Taking a scalar product of this equation with ϕ_j, making use of the fact that $\|\phi_j\|_{L^2(\mathbf{p})} = 1$, and then forming the real and imaginary parts of the result, we obtain

$$|\mathrm{Im}\,\lambda_j(k\varepsilon)| = \left|\int d_3 p\phi_j^*(\mathbf{p}) \frac{i\mathbf{kp}}{p_0} \phi_j(\mathbf{p})\right| \leqslant |\mathbf{k}\varepsilon| \tag{5.2}$$

$$\mathrm{Re}\,\lambda_j(k\varepsilon) = \int d_3 p\phi_j^*(\mathbf{p})[-\nu(\mathbf{p}) + K]\,\phi_j(\mathbf{p}) \tag{5.3}$$

As $-\nu(\mathbf{p}) + K$ is nonpositive definite, we see that $\mathrm{Re}[\lambda_j(k\varepsilon)] \leqslant 0$. Moreover, for $k \neq 0$, if we assume that $\phi_j(k\varepsilon) = \sum_{l=1}^5 \alpha_{jl}(\mathbf{k}\varepsilon)\phi_1^0$, where $\phi_1^0 \in \mathrm{Ker}[-\nu(\mathbf{p}) + K]$, then Eq. (5.1) takes the form

$$\left[-\frac{i\varepsilon\mathbf{kp}}{p_0} - \lambda_j(k\varepsilon)\right] \sum_{l=1}^{5} \alpha_{jl}(\mathbf{k}\varepsilon)\phi_j^0 = 0 \tag{5.4}$$

The only solution of Eq. (5.4) has the form $\alpha_{jl}(\mathbf{k}\varepsilon) = 0$ for all j, $l = 1,..., 5$.

We see that for $k\varepsilon \neq 0$, $\phi_j = \sum_{l=1}^5 \alpha_{jl}(\mathbf{k}\varepsilon)\phi_j^0$ cannot be an eigenfunction of the operator $B_{k\varepsilon}$. From the fact that $-\nu(\mathbf{p}) + K$ is a strongly negative operator for $\phi \notin \mathrm{Ker}(-\nu(\mathbf{p}) + K)$ we can conclude that $\exists C_j > 0$ such that

$$\mathrm{Re}[\lambda_j(k\varepsilon)] \leqslant -C_j \tag{5.5}$$

for all λ_j from the discrete spectrum.

For a more detailed description of the $\sigma(B_{k\varepsilon})$ for large $k\varepsilon$ we need the following lemma.

Lemma 5.1. For $\mathrm{Re}\,\lambda > -1$, $\varepsilon \in]0, 1]$,

$$\lim_{k \to \infty} \|KA_{\lambda k\varepsilon}^{-1}\| = 0 \tag{5.6}$$

where

$$A_{\lambda k\varepsilon}^{-1} = \left[-\nu(\mathbf{p}) - \frac{i\varepsilon\mathbf{kp}}{p_0}\right]^{-1} \tag{5.7}$$

Proof. We make use of the classical results that a compact operator transforms weakly convergent sequences into norm converging ones. For every $\mathbf{k} \in \mathbf{R}^3$ and λ such that $\operatorname{Re} \lambda > -1$ the operator $KA_{\lambda k \varepsilon}^{-1}$ is compact as a composition of a compact and bounded operators. We assume then that there exist sequences k_n such that $k_n \to \infty$ as $n \to \infty$ and ϕ^{k_n} with $\|\phi^{k_n}\| = 1$ such that

$$\|KA_{\lambda k_n \varepsilon}^{-1} \phi^{k_n}\|_{L^2(\mathbf{p})} \nrightarrow 0 \tag{5.8}$$

It follows then that $A_{\lambda k_n \varepsilon}^{-1} \phi^{k_n} \nrightarrow 0$ weakly, but we have the estimates

$$\forall_{\psi \in L^2(\mathbf{p})} (\psi, A_{\lambda k_n \varepsilon}^{-1} \phi^{k_n})_{L^2(\mathbf{p})} \leqslant \left[\int |\psi|^2 \, |A_{\lambda k_n \varepsilon}|^{-2} \, d_3 p \right]^{1/2} \tag{5.9}$$

and according to the Lebesgue theorem,[22] we see that

$$\lim_{n \to \infty} \int d_3 p \, |\psi|^2 \, |A_{\lambda k_n \varepsilon}|^{-2} = 0 \tag{5.10}$$

This is a contradiction, which proves the lemma. ∎

After these preparations, we can prove the following

Theorem 5.2. For every $0 < \beta < 1$ there exists k_β such that for $k\varepsilon > k_\beta$ the half-space $\operatorname{Re} \lambda > -\beta$ is free from eigenvalues.

Proof. We transform Eq. (5.1),

$$[A_{\lambda k \varepsilon} + K] \phi_\lambda = 0, \qquad \phi_\lambda \in D(A_{\lambda k \varepsilon})$$

into the following equivalent one,

$$KA_{\lambda k \varepsilon}^{-1} \psi_\lambda = \psi_\lambda, \qquad \psi_\lambda \in L^2(\mathbf{p}) \tag{5.11}$$

Calculating now the norm of both sides of this equation, we find that

$$\|KA_{\lambda k \varepsilon}^{-1} \psi_\lambda\|_{L^2(\mathbf{p})} = \|\psi_\lambda\|_{L^2(\mathbf{p})} \tag{5.12}$$

but we have that

$$\|KA_{\lambda k \varepsilon} \psi_\lambda\|_{L^2(\mathbf{p})} \leqslant \|KA_{\lambda k \varepsilon}\| \, \|\psi_\lambda\|_{L^2(\mathbf{p})}$$

and according to Lemma 5.1, $\lim_{k \to \infty} \|KA_{\lambda k \varepsilon}\| = 0$. Thus, we can choose sufficiently large k_β such that for $k\varepsilon > k_\beta$, $\|KA_{\lambda k \varepsilon}\| < 1$ and Eq. (5.12) has no solution in $L^2(\mathbf{p})$. ∎

The dependence of the constant k_β on β follows from the fact that $\|KA_{\lambda k \varepsilon}\|$ depends on the lower bound on $\operatorname{Re} \lambda$.

We have shown that for $k \neq 0$ for each λ_i belonging to the discrete spectrum of the operator $B_{k\varepsilon}$ there exists $C_i > 0$ such that $\operatorname{Re} \lambda_i(k\varepsilon) \leqslant -C_i$. We show now a much stronger bound:

Proposition 5.1. For all $k_\alpha > 0$ there exists $C_\alpha > 0$ such that for all $k\varepsilon \in [k_\alpha, k_\beta]$, $\sigma(B_{k\varepsilon}) \cap \{\lambda\colon \operatorname{Re} \lambda > -C_\alpha\} = \varnothing$.

Proof. Assume the contrary, i.e., that there exist $k_0 \in [k_a, k_\beta]$ and sequences ϕ^n and λ_n such that

$$\lim_{n \to \infty} \operatorname{Re}[\lambda_n(k_0\varepsilon)] = 0$$

In such a case we have also that $\exists \gamma$ such that $|\gamma| \leqslant k_0 \varepsilon$ and

$$\lim_{n \to \infty} [\operatorname{Im} \lambda_n(k\varepsilon)] = \gamma$$

This follows from the fact that $\operatorname{Im}[\lambda_n(k_0\varepsilon)] \in [-k_0\varepsilon, k_0\varepsilon]$ and can be considered as a sequence of the elements of the compact set and we can, then, dropping to the subsequence if necessary, choose a convergent subsequence. This in fact shows that the spectrum of the operator $B_{k_0\varepsilon}$ has an accumulation point at $\lambda = i\gamma$. Such a result contradicts Lemma 3.2, which says that for all $\mathbf{k} \in \mathbf{R}^3$ the eigenvalues of the operator $B_{k\varepsilon}$ can accumulate only at the boundary $\partial\Gamma$ of the continuous part of the spectrum. ∎

From Proposition 5.1 and Theorem 5.2 we have the following simple result.

Corollary 5.1. For all $k > 0$ there exists $C > 0$ such that $\sigma(k\varepsilon) \cap \{\lambda\colon \operatorname{Re} \lambda > -C\} = \varnothing$.

Proof. Theorem 5.2 ensures that for $k\varepsilon > k_\beta \varepsilon$, $\sigma(B_{k\varepsilon}) \cap \{\lambda\colon \operatorname{Re} \lambda > -\beta\} = \varnothing$. According to Proposition 5.1, if we take $k > k_\alpha$, then it is enough to choose $C = \min\{\beta, C_\alpha\}$. ∎

6. THE RELATIVISTIC BOLTZMANN SEMIGROUP

We can now formulate and prove the main theorem providing an explicit form of the semigroup which solves the LRBE.

Theorem 6.1. Assume that $\sigma(g, \Theta)$ satisfies (2.10a), (2.10b). Then:

(a) The operator

$$B_\varepsilon = -\frac{\mathbf{p}}{p_0}\nabla + \frac{1}{\varepsilon} L$$

for all $\varepsilon \in]0, 1]$ generates a strongly continuous contraction semigroup on $H_l(\mathbf{r}, \mathbf{p})$, $l \geqslant 0$, given explicitly as

$$\exp(tB_\varepsilon) f(\mathbf{r}, \mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int d_3k \exp(i\mathbf{kr}) \times \exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p}) \tag{6.1}$$

(b) The operators $B_{k\varepsilon}$ for all $\mathbf{k} \in \mathbf{R}^3$ generate a strongly continuous contraction semigroups on $L^2(\mathbf{p})$ and $\exists \delta_1, \beta_1, \beta_2$ such that the following are fulfilled for all $f \in L^2(\mathbf{p})$:

(i) For $k\varepsilon < \delta_1$

$$\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f = \sum_{j=1}^{5} \exp\left[\frac{t}{\varepsilon} \lambda_j(k\varepsilon)\right] (e_j(-\mathbf{k}\varepsilon), f)_{L^2(\mathbf{p})}\, e_j(\mathbf{k}\varepsilon) + \exp\left(\frac{t}{\varepsilon} A_{k\varepsilon}\right) f + \exp\left(\frac{t}{\varepsilon} \beta_1\right) Z_1\left(\mathbf{k}\varepsilon, \frac{t}{\varepsilon}\right) f \tag{6.2}$$

where $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ are eigenvalues and eigenfunctions of the operator $B_{k\varepsilon}$ and they have analytic expansions

$$\lambda_j(k\varepsilon) = \sum_{n=1}^{2} \lambda_{jn}(ik\varepsilon)^n + o(k\varepsilon)^2 \tag{6.3a}$$

$$e_j(\mathbf{k}\varepsilon) = \sum_{n=0}^{2} e_{jn}(\mathbf{k}/k)(ik\varepsilon)^n + o(k\varepsilon)^2 \tag{6.3b}$$

λ_{jn} are constants, with $\lambda_{j2} > 0$, e_{jn} are functions of $\mathbf{k}/k$ only, and

$$e_i(-\mathbf{k}\varepsilon), e_j(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})} = \delta_{ij}, \qquad i, j = 1,..., 5. \tag{6.4}$$

(ii) For $k\varepsilon > \delta_1$

$$\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f = \exp\left(\frac{t}{\varepsilon} A_{k\varepsilon}\right) f + \exp\left(-\frac{t}{\varepsilon} \beta_2\right) Z_2\left(\mathbf{k}\varepsilon, \frac{t}{\varepsilon}\right) f \tag{6.5}$$

where

$$A_{k\varepsilon} = -v(\mathbf{p}) + \frac{i\mathbf{k}\varepsilon\mathbf{p}}{p_0} \tag{6.6}$$

$$Z_j(\mathbf{k}, t) f(\mathbf{k}) = \lim_{\gamma \to \infty} \frac{1}{2\pi i} \int_{-i\gamma}^{i\gamma} e^{it\gamma} Z(k, -\beta_j + i\gamma) f(\mathbf{k})\, d\gamma \tag{6.7}$$

where, for $\lambda \in C$

$$Z(\mathbf{k}, \lambda) = (\lambda - A_{k\varepsilon})^{-1} [I - K(\lambda - A_{k\varepsilon})^{-1}]^{-1} K(\lambda - A_{k\varepsilon})^{-1} \tag{6.8}$$

and

$$\|Z(\mathbf{k}, t) f(\mathbf{k}, \mathbf{p})\|_{L^2(\mathbf{p})} \leqslant C \|f(\mathbf{k}, \mathbf{p})\|_{L^2(\mathbf{p})} \tag{6.9}$$

The constant C is independent of k and t.

Proof. The first part of this theorem was given in I. The proof follows closely that given for the nonrelativistic Boltzmann equation by Nishida and Imai[13] and Elis and Pinsky[12] and can be easily extended to the case $\varepsilon \in]0, 1]$. I present here the proof of part (b).

We use the contour integral representation of the semigoup $\exp[(t/\varepsilon) B_{k\varepsilon}]$ valid for any $f \in D(B_{k\varepsilon})$,

$$\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p}) = \frac{1}{2\pi i} \int_C \exp\left(\frac{zt}{\varepsilon}\right) \times (z - B_{k\varepsilon})^{-1} f(\mathbf{k}, \mathbf{p})\, dz \tag{6.10}$$

where C is a contour which lies to the right of $\sigma(B_{k\varepsilon})$ and extends from $-i\infty$ to $i\infty$. For $k\varepsilon < \delta_1$, where we take δ_1 small enough to apply Theorem 4.2, we can make the following transformation of the contour integration in Eq. (6.10):

$$\begin{aligned} \frac{1}{2\pi i} \int_C \exp\left(\frac{zt}{\varepsilon}\right) (z - B_{k\varepsilon})^{-1} f\, dz &= -\frac{1}{2\pi i} \int_C \exp\left(\frac{zt}{\varepsilon}\right) (z - B_{k\varepsilon})^{-1} f\, dz \\ &+ \frac{1}{2\pi} \lim_{\gamma \to \infty} \int_{-\beta_1 - i\gamma}^{-\beta_1 + i\gamma} \exp\left[-(\beta_1 + i\gamma) \frac{t}{\varepsilon} \right] (-\beta_1 + i\gamma - B_{k\varepsilon})^{-1} f\, d\gamma \end{aligned} \tag{6.11}$$

$C_\delta = \{z \in C\colon z = \delta e^{i\phi},\ \phi \in [0, 2\pi[\}$ with $\delta \in]\frac{1}{3}\nu_1, \frac{2}{3}\nu_1[$ contains the first five eigenvalues of the operator $B_{k\varepsilon}$. We can choose also $\beta_1 \in]\frac{1}{3}\nu_1, \frac{2}{3}\nu_1[$, and, as shown in Theorem 4.2, $\sigma(B_{k\varepsilon}) \cap \{z\colon z = -\beta + i\gamma;\ \beta = \beta_1,\ \gamma \in R\} = \varnothing$. We see then that for a given $k\varepsilon < \delta_1$ the integral over C_δ in Eq. (6.11) splits into n_i integrals over small circles, each encircling only one eigenvalue of $B_{k\varepsilon}$; $n_i \leqslant 5$ is the number of distinct eigenvalues of $B_{k\varepsilon}$ in C_δ. These integral can be easily performed and we obtain

$$\frac{1}{2\pi i} \int_C \exp\left(\frac{zt}{\varepsilon}\right) (z - B_{k\varepsilon})^{-1} f\, dz = \sum_{l=1}^{n_i} \exp\left[\frac{\lambda_1(k\varepsilon) t}{\varepsilon}\right] P_l(\mathbf{k}\varepsilon) f \tag{6.12}$$

where $P_l(\mathbf{k}\varepsilon)$ are the spectral projectors on the subspace corresponding to the $\lambda_l(k\varepsilon)$. It follows from Theorem 4.2 that these projectors can be represented as

$$P_l(\mathbf{k}\varepsilon) f = \sum_{j=1}^{m_j} (e_{l_j}(-\mathbf{k}\varepsilon), f)_{L^2(\mathbf{p})}\, e_{l_j}(\mathbf{k}\varepsilon)$$

m_j is the multiplicity of $\lambda_j(k\varepsilon)$.

We can write

$$-(1/2\pi i)\int_{C_\delta} \exp(zt/\varepsilon)(z-B_{k\varepsilon})^{-1} f(\mathbf{k},\mathbf{p})\, dz$$
$$= \sum_{j=1}^{5} \exp[\lambda_j(k\varepsilon)t/\varepsilon](e_j(-\mathbf{k}\varepsilon), f)_{L^2(\mathbf{p})}\, e_j(\mathbf{k}\varepsilon) \tag{6.13}$$

with

$$(e_j(-\mathbf{k}\varepsilon), e_l(\mathbf{k}\varepsilon))_{L^2(\mathbf{p})} = \delta_{jl}$$

We have shown that $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ are analytic functions of $k\varepsilon$ near zero; thus, taking δ_1 sufficiently small, we fulfill (6.3a), (6.3b).

In the second integral in Eq. (6.11) with the help of the operator identity $(z-A-B)^{-1} = (z-A)^{-1} + (z-A)^{-1} B(z-A-B)^{-1}$ we obtain

$$(1/2\pi)\int_{-\beta_1-i\gamma}^{-\beta_1+i\gamma} \exp[(-\beta_1+i\gamma)t/\varepsilon](-\beta_1+i\gamma-B_{k\varepsilon})^{-1} f(\mathbf{k},\mathbf{p})\, dz$$
$$= (1/2\pi)\int_{-\beta_1-i\gamma}^{-\beta_1+i\gamma} \exp[-(\beta_1+i\gamma)t/\varepsilon[(-\beta_1+i\gamma-A_{k\varepsilon})^{-1} f\, d\gamma$$
$$+ (1/2\pi)\exp(-\beta_1 t/\varepsilon)\int_{-\infty}^{\infty} Z(-\beta_1+i\gamma, k\varepsilon) f\, d\gamma \tag{6.14}$$

where

$$Z(z, \mathbf{k}\varepsilon) = (z-A_{k\varepsilon})^{-1}\,[I-K(z-A_{k\varepsilon})^{-1}]^{-1}\, K(z-A_{k\varepsilon})^{-1} \tag{6.15}$$

The reason for such a splitting is that the first integral on the rhs of E. (6.15) can be explicitly done, with the result

$$(1/2\pi)\int_{-\infty}^{\infty} \exp[(-\beta_1+i\gamma)t/\varepsilon)](-\beta_1+i\gamma-A_{k\varepsilon})^{-1} f\, d\gamma$$
$$= \exp(A_{k\varepsilon}t/\varepsilon) f$$
$$\equiv \exp\{[-\nu(\mathbf{p})/\varepsilon + i\mathbf{k}\mathbf{p}/p_0]t\} f(\mathbf{k},\mathbf{p}) \tag{6.16}$$

The norm of the operator $Z(z, \mathbf{k})$ can be estimated for $z = -\beta + i\gamma \notin \sigma(B_{k\varepsilon})$ as

$$\|Z(z, \mathbf{k})\| \leqslant \left[(\beta - 1)^2 + \left(\gamma - \frac{\mathbf{kp}\varepsilon}{p_0}\right)^2\right]^{-1}$$
$$\times \|K\| \, \|[I - K(z - A_{k\varepsilon})^{-1}]^{-1}\| \tag{6.17}$$

The operator $K(z - A_{k\varepsilon})^{-1}$ is compact for $z \notin \sigma(A_{k\varepsilon})$ as a product of a compact and bounded operators. Thus, $[I - K(z - A_{k\varepsilon})^{-1}]^{-1}$ is bounded provided that 1 is not an eigenvalue of the operator $K(z - A_{k\varepsilon})^{-1}$. This condition is equivalent to the requirement that $z \notin \sigma(K + A_{k\varepsilon}) \equiv \sigma(B_{k\varepsilon})$. Note for completeness that if $z \notin \sigma(B_{k\varepsilon})$, then also $z \notin \sigma(A_{k\varepsilon})$. Denoting the $\|K\| \, \|[I - K(z - A_{k\varepsilon})^{-1}]^{-1}\|$ by C', we obtain

$$\|Z(z, \mathbf{k}\varepsilon)\| \leqslant C' \left[(\beta - 1)^2 + \left(\gamma - \frac{\mathbf{kp}\varepsilon}{p_0}\right)^2\right]^{-1} \tag{6.18}$$

We see then that

$$\int_{-\infty}^{\infty} \exp(it\gamma/\varepsilon) \, \|Z(-\beta_1 + i\gamma, k\varepsilon) f(\mathbf{k}, \mathbf{p})\|_{L^2(\mathbf{p})} \, d\gamma$$

is absolutely convergent for $\beta_1 \in]\frac{1}{3}\nu_1, \frac{2}{3}\nu_1[$ and if we define

$$Z_1(t/\varepsilon, \mathbf{k}\varepsilon) f(\mathbf{k}, \mathbf{p}) = \int_{-\infty}^{\infty} Z(-\beta_1 + i\gamma, \mathbf{k}\varepsilon) f(\mathbf{k}, \mathbf{p}) \, d\gamma \tag{6.19}$$

then

$$\|Z_1(t/\varepsilon, \mathbf{k}\varepsilon) f\|_{L^2(\mathbf{p})} \leqslant C \, \|f\|_{L^2(\mathbf{p})} \tag{6.20}$$

with constant

$$C \geqslant C' \int_{-\infty}^{\infty} d\gamma \left[(\beta - 1)^2 + \left(\gamma - \frac{\mathbf{kp}\varepsilon}{p_0}\right)^2\right]^{-1}$$

which can be chosen independent of k, t, and ε.

(ii) For $k\varepsilon > \delta_1$ we know that $\exists \mu > 0$ such that for all $\beta < \mu$, $\sigma(B_{k\varepsilon}) \cap \{z: -\beta + i\gamma, \gamma \in R\} = \varnothing$. Choosing $0 < \beta_2 < \mu$, we have

$$(1/2\pi i) \int_C \exp(zt/\varepsilon)(z - B_{k\varepsilon})^{-1} f(\mathbf{k}, \mathbf{p}) \, dz$$
$$= (1/2\pi) \int_{-\infty}^{\infty} \exp[-(\beta_2 + i\gamma) t/\varepsilon](-\beta_2 + i\gamma - A_{k\varepsilon})^{-1} f \, d\gamma$$
$$+ \exp(\beta_2 t/\varepsilon)(1/2\pi) \int_{-\infty}^{\infty} \exp(i\gamma t/\varepsilon) \, Z(-\beta_2 + i\gamma) f \, d\gamma \tag{6.21}$$

Repeating now the same arguments as above, we obtain

$$\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p}) = \exp\left(\frac{t}{\varepsilon} A_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p}) + Z_2\left(\frac{t}{\varepsilon}, \mathbf{k}\varepsilon\right) f(\mathbf{k}, \mathbf{p}) \tag{6.22}$$

$$\|Z_2(t/\varepsilon, \mathbf{k}\varepsilon) f\|_{L^2(\mathbf{p})} \leqslant C \, \|f\|_{L^2(\mathbf{p})} \tag{6.23}$$

We have constructed a representation of the semigroup $\exp[(t/\varepsilon) B_{k\varepsilon}]$ for $f \in D(B_{k\varepsilon})$ a dense subset of $L^2(\mathbf{p})$ and this semigroup is a contraction semigroup; thus, we can extend these results on the whole $L^2(\mathbf{p})$. ▮

With Theorem 6.1 we prove the decay estimate for the solution to the LRBE with initial data $f \in H_l(\mathbf{r}, \mathbf{p})$. These solutions are defined as

$$f_\varepsilon(\mathbf{r}, \mathbf{p}, t) = \exp(tB_\varepsilon) f(\mathbf{r}, \mathbf{p}) \tag{6.24}$$

From the contraction property of the semigroup $\exp(tB_\varepsilon)$ we see that

$$\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t)\|_l \leqslant \|f(\mathbf{r}, \mathbf{p})\|_l \tag{6.25}$$

More detailed information is provided by the following.

Theorem 6.2. For all $\varepsilon \in]0, 1]$ the solution to the LRBE with initial data $f(\mathbf{r}, \mathbf{p})$ has the following decay estimates:

(i) For $f \in H_l(\mathbf{r}, \mathbf{p})$

$$\lim_{t \to \infty} \|\exp(tB_\varepsilon) f\|_l = 0 \tag{6.25a}$$

(ii) For $f \in H_l(\mathbf{r}, \mathbf{p}) \cap L^1(\mathbf{r}, L^2(\mathbf{p}))$

$$\|\exp(tB_\varepsilon) f\|_l \leqslant C_\varepsilon(\|f\|_l + \|f\|_{L^1(r, L^2(\mathbf{p}))})/(1+t)^{3/4} \tag{6.25b}$$

(iii) For $f \in H_l(\mathbf{r}, \mathbf{p})$, $rf \in L^1(\mathbf{r}, L^2(\mathbf{r}, \mathbf{p}))$, and, $\int d_3r \, d_3p f \psi_j = 0$, $j = 1,..., 5$, $\psi_j \in \mathrm{Ker}(L)$, we have

$$\|\exp(tB_\varepsilon)\|_l \leqslant C_\varepsilon C \, \|f\|_l + \sup_{\mathbf{k}} \|f(\mathbf{k}, \circ)\|_{L^2(\mathbf{p})})/(1+t)^{5/4} \tag{6.25c}$$

where the constant C_ε depends on ε only.

Proof. According to the Theorem 6.1, we can write for $f \in H_l(\mathbf{r}, \mathbf{p})$

$$\begin{aligned}
&\|\exp(tB_\varepsilon) f\|_l^2 \\
&= \iint d_3k\, d_3p (1+k^2)^l \left|\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p})\right|^2 \\
&= \int d_3 p \left[\int_{k\varepsilon<\delta_1} (1+k^2)^l \left|\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p})\right|^2 d_3k \right. \\
&\quad \left. + \int_{k\varepsilon>\delta_1} (1+k^2)^l \left|\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p})\right|^2 d_3k \right] = I + I_3
\end{aligned}$$

with

$$\begin{aligned}
I = \int d_3 p &\left[(1+k^2)^l \left\{ \left| \sum_{j=1}^{5} \exp\left[\frac{t}{\varepsilon} \lambda_j(k\varepsilon)\right] \right.\right.\right. \\
&\times (e_j(-\mathbf{k}\varepsilon), f(\mathbf{k}, \mathbf{p}))_{L^2(\mathbf{p})}\, e_j(\mathbf{k}\varepsilon) \\
&+ \exp\left(\frac{t}{\varepsilon} A_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p}) + \exp\left(-\frac{t}{\varepsilon}\beta_1\right) \\
&\left.\left.\left. \times Z_1\left(\frac{t}{\varepsilon}, \mathbf{k}\varepsilon\right) f(\mathbf{k}, \mathbf{p}) \right|^2 \right\} \right] = I_1 + I_2
\end{aligned}$$

I_2 can be estimated as follows:

$$I_2 \leqslant e^{-2t/\varepsilon} \|f\|_1^2 + e^{-2(t/\varepsilon)\beta_1} C^2 \|f\|_1^2 + 2Ce^{-(t/\varepsilon)(1+\beta_1)} \|f\|_1^2$$

For I_3 we can write

$$\begin{aligned}
I_3 &= \int d_3 p \int_{k\varepsilon>\delta_1} d_3k (1+k^2)^l \left|\exp\left(\frac{t}{\varepsilon} B_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p})\right|^2 \\
&\quad + \int d_3 p \int_{k\varepsilon>\delta_1} d_3k (1+k^2)^l \left|\exp\left(\frac{t}{\varepsilon} A_{k\varepsilon}\right) f(\mathbf{k}, \mathbf{p}) \right. \\
&\quad \left. + \exp\left(-\frac{t}{\varepsilon}\beta_2\right) Z_2\left(\frac{t}{\varepsilon}, \mathbf{k}\varepsilon\right) f(\mathbf{k}, \mathbf{p})\right|^2 \\
&\leqslant \exp\left(-\frac{2}{\varepsilon} t\right) \|f\|_l^2 + \exp\left(-2\frac{t}{\varepsilon}\beta_2\right) C^2 \|f\|_l^2 \\
&\quad + 2C \exp\left[-\frac{t}{\varepsilon}(1+\beta_2)\right] \|f\|_l^2
\end{aligned}$$

We see that the main problem is to estimate the I_1. To this end, we observe that

$$\lim_{t \to \infty} \exp\left[2\frac{t}{\varepsilon}\lambda_j(k\varepsilon)\right] = 0 \qquad \text{for} \quad \forall k \neq 0$$

and

$$\int_{k\varepsilon < \delta_1} d_3k(1+k^2)^l \sum_{j=1}^{5} (e_j(-\mathbf{k}\varepsilon), f(\mathbf{k}, \mathbf{p}))_{L^2(\mathbf{p})} \leqslant \|f\|_l < \infty$$

Applying then the Lebesgue theorem[22] to I_1, we obtain

$$\lim_{t \to \infty} I_1 = 0$$

and it follows that

$$\lim_{t \to \infty} \|\exp(tB_\varepsilon)f\|_l \leqslant \lim_{t \to \infty} (I_1 + I_2 + I_3) = 0$$

This proves (i). A function $f \in H_l(\mathbf{k}, \mathbf{p})$ can be concentrated at $k = 0$ and without additional information concerning the behavior of $(e_j(-\mathbf{k}\varepsilon), f)_{L^2(\mathbf{p})}$ for small k we cannot obtain stronger bounds on the decay of $f(\mathbf{r}, \mathbf{p}, t)$. Such an assumption is provided by (ii). If $f(\mathbf{r}, \mathbf{p}) \in H_l(\mathbf{r}, \mathbf{p}) \cap L^1(\mathbf{r}, L^2(\mathbf{p}))$, it means that $\sup_k \sum_{j=1}^5 (e_j(-\mathbf{k}\varepsilon), f)_{L^2(\mathbf{p})} < \infty$. This allows for improving the estimate for I_1. Denoting

$$-\Lambda_2(k\varepsilon)^2 = \max_{k\varepsilon < \delta_1} \{\mathrm{Re}[\lambda_j(k\varepsilon)], j = 1, \ldots, 5\}$$

we obtain

$$\begin{aligned}
&\int_{k\varepsilon < \delta_1} d_3k(1+k^2)^l \sum_{j=1}^{5} \exp\{2\,\mathrm{Re}[\lambda_j(k\varepsilon)]\,t/\varepsilon\} \\
&\qquad \times |(e_j(-\mathbf{k}\varepsilon), f(\mathbf{k}, \mathbf{p})|^2 \\
&\quad \leqslant \int_{k\varepsilon < \delta_1} d_3k(1+k^2)^l \exp(-2\Lambda_2 k^2 \varepsilon t)\, \|f(\mathbf{k}, \mathbf{p})\|^2_{L^2(\mathbf{p})} \\
&\quad \leqslant \int_{k\varepsilon < \delta_1} d_3k(1+k^2)^l \exp(-2\Lambda_2 k^2 \varepsilon t) \\
&\qquad \times \sup_k [(1+k^2)^l\, \|f(\mathbf{k}, \mathbf{p})\|^2_{L^2(\mathbf{p})}] \\
&\quad \leqslant C(1+t\varepsilon)^{-3/2}\, \|f(\mathbf{r}, \mathbf{p})\|^2_{L^1(\mathbf{r}, L^2(\mathbf{p}))}
\end{aligned}$$

For $\varepsilon \neq 0$ we can write

$$C(1+t\varepsilon)^{-3/2} \leqslant C_\varepsilon(1+t)^{-3/2}$$

We can write also $I_1 + I_2 \leqslant C(1+t)^{-3/2} \|f\|_l$ and this gives (ii).

It follows from Theorem 6.1 that

$$e_j(k\varepsilon) = \sum_{n=0}^{2} e_{jn}(ik\varepsilon)^n + o[(k\varepsilon)^2]$$

The assumption made in (iii) ensures that

$$(e_j(-\mathbf{k}\varepsilon), f(\mathbf{k}, \mathbf{p}))_{L^2(\mathbf{p})}|_{k=0} = 0$$

$$\sup_{\mathbf{k}} |\nabla_{\mathbf{k}}(e_{j0}, f(\mathbf{k}, \mathbf{p}))_{L^2(\mathbf{p})}| < \infty$$

We see then that for $k\varepsilon < \delta_1$,

$$(e_j(-k\varepsilon), f(\mathbf{k}, \mathbf{p}))_{L^2(\mathbf{p})} \leqslant k\varepsilon g(\mathbf{k})$$

where $\sup_{\mathbf{k}} |g(\mathbf{k})| < \infty$. It is possible then to improve the estimate of I_1 and we obtain

$$\begin{aligned} I_1 &\leqslant C \int_{k\varepsilon < \delta_1} d_3k(1+k^2)^l \exp(-2\Lambda_2 k^2 \varepsilon t)\, k^2\varepsilon^2\, |g(\mathbf{k})|^2 \\ &\leqslant C\varepsilon^2 \int_{k\varepsilon < \delta_1} d_3k\, k^2 \exp(-2\Lambda_2 k^2 \varepsilon t) \\ &\quad \times [\sup_{k\varepsilon < \delta_1} (1+k^2)^l\, |g(\mathbf{k})|^2] \\ &\leqslant C_\varepsilon(1+t)^{-5/2} [\sup_{\mathbf{k}} \|f(\mathbf{k}, \mathbf{p})\|_{L^2(\mathbf{p})}]^2 \end{aligned}$$

As $I_2 + I_3 \leqslant C(1+t)^{-5/2} \|f\|_l^2$, we obtain (iii). ∎

Note that the decay estimates for the relativistic Boltzmann equation are in fact very similar to those obtained for the nonrelativistic Boltzmann equation by Nishida and Imai.[13]

7. THE HYDRODYNAMICAL APPROXIMATION

For sufficiently long times $t \gg \max\{\varepsilon\beta_1^{-1}, \varepsilon\beta_2^{-1}\}$ the dominant part of the solution to the LRBE has the following form:

$$\begin{aligned} f_\varepsilon(\mathbf{k}, \mathbf{p}, t) = \sum_{j=1}^{5} \exp\left[\frac{t}{\varepsilon}\lambda_j(k\varepsilon)\right] \\ \times (e_j(-k\varepsilon), f(\mathbf{k}, \mathbf{p}, 0))_{L^2(\mathbf{p})}\, e_j(\mathbf{k}\varepsilon) \end{aligned} \tag{7.1}$$

From the perturbation expansions of $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ (Appendix A) it follows that up to terms of second order in $k\varepsilon$ we can write

$$\lambda_j(k\varepsilon) = ik\varepsilon\lambda_{j1} - \lambda_{j2}k^2\varepsilon^2 + o[(k\varepsilon)^2] \tag{7.2a}$$

$$e_j(\mathbf{k}\varepsilon) = e_{j0} + ik\varepsilon \sum_{l=1}^{5} B_{lj}e_{l0} + ik\varepsilon\psi_j + o(k\varepsilon) \tag{7.2b}$$

All $\{\psi_j\}_{j=1}^5$ are orthogonal to $N_0 = \{e_{j0}\}_{j=1}^5$. Explicit expressions for the coefficients in the expansions (7.2) have the form

$$\lambda_{j1} = -\left(e_{j0}, \frac{\mathbf{pk}}{p_0 k}\, e_{j0}\right) \tag{7.3a}$$

$$\lambda_{j2} = -\left(e_{j0}, \frac{\mathbf{kp}}{p_0 k}\,(QLQ)^{-1}\,\frac{\mathbf{kp}}{p_0 k}\, e_{j0}\right) \tag{7.3b}$$

$$B_{j1}^1 = \begin{cases} -(\lambda_{j1} - \lambda_{l1})^{-1}\,(e_{j0}, (\mathbf{pk}/p_0 k)(QLQ)^{-1}\,(\mathbf{kp}/p_0 k)e_{l0}), & i \neq 1, \lambda_{j1} \neq \lambda_{l1} \\ 0, & \text{otherwise} \end{cases} \tag{7.4}$$

Q is a projector on the subspace orthogonal to N_0. These eigenfunctions e_{j0} depend on p and the scalar product $(\circ, \circ)$ is understood as the scalar product in $L^2(\mathbf{p})$.

We define a set of auxiliary hydrodynamic variables $n_i(r, t)$ with a help of the eigenvectors e_{j0} as

$$n_i(\mathbf{r}, t) = (e_{j0}, f(\mathbf{r}, \mathbf{p}, t)) \tag{7.5}$$

We see that

$$\begin{aligned} f_\varepsilon(\mathbf{k}, \mathbf{p}, t) = \sum_{j=1}^{5} \exp\left[\frac{t}{\varepsilon}\lambda_j(k\varepsilon)\right]\Bigg\{ & n_j(\mathbf{k}, 0) + ik\varepsilon \\ & \times \sum_{l=1}^{5} B_{j1}^1 n_l(\mathbf{k}, 0) + ik\varepsilon(\psi_j, f(\mathbf{k}, \mathbf{p}, 0)) \\ & \times \left[e_{j0} + ik\varepsilon \sum_{n=1}^{5} B_{jn}^1 e_{j0} + ik\varepsilon\psi_j\right]\Bigg\} + o(k\varepsilon) \end{aligned} \tag{7.6}$$

In order to introduce the hydrodynamic approximation, we define a set of differential equations for variables $n_i(\mathbf{r}, t)$, $i = 1,..., 5$,

$$\partial_t n_i(\mathbf{r}, t) = \sum_{j=1}^{5} A_\varepsilon(\nabla)_{ij}\, n_j(\mathbf{r}, t) \tag{7.7}$$

where

$$A_\varepsilon(\nabla)_{nm} = (2\pi)^{-3/2} \int d_3k[\exp(ikr)]\, A_\varepsilon(\mathbf{k})_{nm} \tag{7.8}$$

and

$$A_\varepsilon(\mathbf{k})_{nm} = \begin{cases} (1/\varepsilon)[ik\varepsilon\lambda_{n1} - (k\varepsilon)^2 \lambda_{n2}], & n = m \\ -(\lambda_{n1} - \lambda_{m1})\, B^1_{nm}(\mathbf{k}\varepsilon)^2, & n \neq m \end{cases} \tag{7.9}$$

It is easy to see from the definition of B^1_{ij} that $A_\varepsilon(\mathbf{k})_{ij} = A_\varepsilon(\mathbf{k})_{ji}$ for $i \neq j$. The following proposition describes the behavior of the eigenvalues of the matrix $A_\varepsilon(\mathbf{k})$.

Proposition 7.1. Denote the eigenvalues of the $A_\varepsilon(\mathbf{k})$ as $\Lambda^\varepsilon_j(k)$, $j = 1, \ldots, 5$; then:

(i) For all $\mathbf{k} \in \mathbf{R}^3$ and $\varepsilon \in]0, 1]$

$$\operatorname{Re}[\Lambda^\varepsilon_j(k)] \leqslant 0 \tag{7.10}$$

(ii) $\exists \delta_2 > 0$ such that for $k < \delta_2$ all these $\Lambda^\varepsilon_j(k)$ are analytic functions of k and have expansions

$$\Lambda^\varepsilon_j(k) = ik\lambda_{j1} - k^2\varepsilon\lambda_{j2} + o(k^2\varepsilon) \tag{7.11}$$

with λ_{jn} given by Eq. (7.2a).

(iii) $\exists \beta_3 > 0$ such that for $k > \delta_2$

$$\operatorname{Re}[\Lambda^\varepsilon_j(k)] \leqslant -\beta_3/\varepsilon \tag{7.12}$$

Proof. Observe that the eigenvalue problem for the matrix $A_\varepsilon(k)$ is equivalent to the problem of finding in the space N_0 the eigenvalues of the operator V defined as

$$V = P\left[-\frac{i\mathbf{kp}}{p_0} + k^2 \frac{\mathbf{kp}}{p_0 k}\,\varepsilon(QLQ)^{-1}\frac{\mathbf{kp}}{p_0 k}\right]P \tag{7.13}$$

where P is a projector on the N_0, and $Q = 1 - P$. To see this equivalence, it is enough to note that the matrix representation of the operator V written in the basis $\{e_{j0}\}_{j=1}^5$ is identical to $A_\varepsilon(\mathbf{k})$. The operator $(QLQ)^{-1}$ is not positive and we see immediately that $\operatorname{Re}[\Lambda^\varepsilon_j(k)] \leqslant 0$, which proves (i).

The eigenvalues of the operator $A_\varepsilon(\mathbf{k})$ depend only on k and without loss of generality we can choose $\mathbf{k} \parallel \hat{z}$. The operator V then simplifies to

$$V = P\left[-ik\frac{p_z}{p_0} + k^2\varepsilon\frac{p_z}{p_0}(QLQ)^{-1}\frac{p_z}{p_0}\right]P \tag{7.14}$$

For k small enough we can treat

$$k^2\varepsilon P\left[\frac{p_z}{p_0}(QLQ)^{-1}\frac{p_z}{p_0}\right]P$$

as a small perturbation of the operator $ik(p_z/p_0)$. Applying the Kato–Rellich theorem to this case, we see that there exists $\delta_2 > 0$ such that for $k < \delta_2$, $\Lambda_j^\varepsilon(k)$, $j = 1,..., 5$, are analytic functions of k. Now using the perturbation expansions with $\{e_{j0}\}_{j=1}^5$ taken as eigenfunctions of the operator $P(p_z/p_0)P$, we obtain (ii). We have

$$\operatorname{Ker}\left[P\left(\frac{p_z}{p_0}(QLQ)^{-1}\frac{p_z}{p_0}\right)P\right]\cap N_0 = \{p_0 f_0^{1/2}\}$$

For $k \neq 0$, $V(p_0 f_0^{1/2}) = ikp_z f_0^{1/2}$ and this shows that for $k \neq 0$, $p_0 f_0^{1/2}$ cannot be an eigenfunction of the operator V. We note now that if ϕ_k is an eigenfunction of V for $k \neq 0$ and $\phi_k = p_0 f_0^{1/2} + \psi$ with $(\psi, p_0 f_0^{1/2}) = 0$, then

$$\operatorname{Re}(\phi, V\phi) = k^2\varepsilon\left(\psi, \frac{p_z}{p_0}(QLQ)^{-1}\frac{p_z}{p_0}\psi\right) \leqslant -k^2\varepsilon\gamma(k) \tag{7.15}$$

If $k\varepsilon > \delta_2$, then

$$\gamma(k)\, k^2\varepsilon \geqslant \gamma(k)\, \delta_2^2/\varepsilon = \mu(k)/\varepsilon \tag{7.16}$$

We show now that there exist k_0 and μ_0 such that for $k\varepsilon > k_0$, $\mu(k) > \mu$. For large enough $k\varepsilon$ we can write V in the form

$$V = k^2\varepsilon P\left[-\frac{ip_z}{k\varepsilon p_0} + \frac{p_z}{p_0}(QLQ)^{-1}\frac{p_z}{p_0}\right]P$$

and treat the first term as a small perturbation of the operator

$$V' = P\left[\frac{p_z}{p_0}(QLQ)^{-1}\frac{p_z}{p_0}\right]P$$

The eigenfunction belonging to the zero eigenvalue of V' is $p_0 f_0^{1/2}$. Denoting the corresponding eigenvalue of V by Λ_0, with the help of a perturbation expansion we obtain

$$\Lambda_0 = ik\Lambda_{01} - \frac{1}{\varepsilon}\Lambda_{02} + \frac{1}{k\varepsilon}\Lambda_{03} + o[(k\varepsilon)^{-1}] \tag{7.17}$$

with $\Lambda_{02} > 0$, and for the other four eigenvalues

$$\operatorname{Re}[\Lambda_j(k)] \leqslant -\gamma_j k^2 \varepsilon^2 \leqslant -\frac{\gamma_j}{\varepsilon} k_0^2$$

We see that, taking k_0 sufficiently large, there exists $\mu > 0$ such that for $k\varepsilon > k_0$, $\operatorname{Re}(\Lambda_j^\varepsilon(k)) \leqslant -\mu$.

Choosing now $\beta_3 = \min(\mu, \mu(k); k \in [\delta_2/\varepsilon, k_0/\varepsilon])$, we obtain (iii). ∎

Our aim now is to solve the hydrodynamic equation (7.7) and show that this solution converges to the solution of the LRBE as $t \to \infty$. Let us prove first the following result.

Theorem 7.1. For $n_i \in H^2(\mathbf{r})$, $i = 1,..., 5$, there exist global in time solutions to Eq. (7.7), $n_i(\mathbf{r}, t) \in H^2(\mathbf{r})$. The explicit form of these solutions reads

$$n_i(\mathbf{r}, t) = (2\pi)^{-3/2} \int [\exp(i\mathbf{kr})]\, n_i(\mathbf{k}, t) \tag{7.18}$$

and there exist constants $\delta_2 > 0$ and $\beta_3 > 0$ such that:

(i) For $k\varepsilon < \delta_2$

$$\begin{aligned} n_j(\mathbf{k}, t) = {} & \exp[\Lambda_j^\varepsilon(k)t] \left[n_j(\mathbf{k}, 0) + ik\varepsilon \sum_{i=1}^{5} B_{jl}^1 n_l(\mathbf{k}, 0) \right] \\ & + \sum_{l=1}^{5} \exp[\Lambda_l^\varepsilon(k)t]\, ik\varepsilon B_{jl} n_l(k, 0) \\ & + \sum_{n=1}^{5} \exp[\Lambda_j^\varepsilon(k)t]\, C_{jn}^\varepsilon(\mathbf{k}, t)\, n_n(\mathbf{k}, t) \end{aligned} \tag{7.19}$$

where

$$\Lambda_j^\varepsilon(k) = ik\lambda_{j1} - k^2\varepsilon\lambda_{j2} + o(k^2\varepsilon)$$

and

$$|C_{ij}^\varepsilon(\mathbf{k}, t)| \leqslant Ck^2\varepsilon$$

(ii) For $k\varepsilon > \delta_2$

$$\begin{aligned} n_j(\mathbf{k}, t) = {} & \sum_{j=1}^{5} \exp[(ik\varepsilon\lambda_{ji} - k^2\varepsilon\lambda_{j2})t]\, n_j(\mathbf{k}, 0) \\ & + \exp(-\beta_3 t/\varepsilon) \sum_{l=1}^{5} \tilde{Z}_{jl}(\mathbf{k}\varepsilon, t/\varepsilon)\, n_l(\mathbf{k}, 0) \end{aligned} \tag{7.20}$$

where

$$\|\tilde{Z}(\mathbf{k}\varepsilon, t/\varepsilon)\| \leqslant C \tag{7.21}$$

Proof. To solve Eq. (7.7), we use the Laplace transform in time and the Fourier transform in r. These result in the following matrix equation:

$$\sum_{j=1}^{5} [zI - A_\varepsilon(\mathbf{k})]_{ij}\, n_j(\mathbf{k}, z) = n_i(\mathbf{k}, 0) \tag{7.22}$$

Its formal solution has the form

$$n_i(\mathbf{k}, z) = \sum_{j=1}^{5} [zI - A_\varepsilon(\mathbf{k})]_{ij}^{-1}\, n_j(\mathbf{k}, 0) \tag{7.23}$$

The time dependence of the solution is determined by

$$n_i(\mathbf{k}, t) = (2\pi i)^{-1} \int_C dz \exp(zt) \times \sum_{j=1}^{5} [zI - A_\varepsilon(\mathbf{k})]_{ij}^{-1}\, n_j(\mathbf{k}, 0) \tag{7.24}$$

where the contour C lies to the right of the spectrum of $A_\varepsilon(\mathbf{k})$ and extends from $-i\infty$ to $i\infty$. For sufficiently small k such that $k\varepsilon < \delta_2$ we can calculate perturbatively $[zI - A_\varepsilon(\mathbf{k})]^{-1}$ and then with Λ_i^ε from Proposition 7.1 we have (i).

For $k\varepsilon > \delta_2$ we split the $A_\varepsilon(\mathbf{k})$ as follows:

$$A_\varepsilon(\mathbf{k}) = C_\varepsilon(k) + D_\varepsilon(\mathbf{k}) \tag{7.25}$$

where

$$C_\varepsilon(k)_{ij} = (ik\lambda_{i1} - k^2\varepsilon\lambda_{i2})\delta_{ij} \tag{7.26}$$

We can write

$$\begin{aligned} &[zI - C_\varepsilon(k) - D_\varepsilon(\mathbf{k})]^{-1} \\ &\quad = [zI - C_\varepsilon(k)]^{-1} \\ &\qquad + [zI - C_\varepsilon(k)]^{-1} \{I - D_\varepsilon(\mathbf{k})[zI - C_\varepsilon(k)]^{-1}\}^{-1} \\ &\qquad \times D_\varepsilon(\mathbf{k})[zI - C_\varepsilon(k)]^{-1} \end{aligned} \tag{7.27}$$

With Eq. (7.27) we can rewrite Eq. (7.24) as

$$
\begin{aligned}
n_i(\mathbf{k}, t) = &\frac{1}{2\pi i}\int_C dz \exp(zt) \sum_{j=1}^{5} [zI - C_\varepsilon(k)]_{ij}^{-1} n_j(\mathbf{k}, 0) \\
&+ \frac{1}{2\pi}\int_{-\beta_3/\varepsilon - i\gamma}^{-\beta_3/\varepsilon + i\gamma} d\gamma \exp\left[\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right)t\right] \\
&\times \sum_{j=1}^{5} \left\{\left[\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right)I - C_\varepsilon(k)\right]^{-1}\right. \\
&\times \left\{I - D_\varepsilon(\mathbf{k})\left[\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right)I - C_\varepsilon(k)\right]^{-1}\right\}^{-1} \\
&\times D_\varepsilon(\mathbf{k})\left.\left[\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right) - C_\varepsilon(k)\right]^{-1}\right\}_{ij} n_j(\mathbf{k}, 0) \qquad (7.28)
\end{aligned}
$$

The first integral can be easily calculated, with the result

$$
\begin{aligned}
&\int_C dz \exp(zt) \sum_{j=1}^{5} [zI - C_\varepsilon(k)]_{lj}^{-1} n_j(\mathbf{k}, 0) \\
&\qquad = \exp[(ik\lambda_{l1} - k^2\varepsilon\lambda_{l2})t]\, n_l(\mathbf{k}, 0) \qquad (7.29)
\end{aligned}
$$

The second integral can be written as

$$
\sum_{j=1}^{5} \exp[-(\beta_3/\varepsilon)t]\, \tilde{Z}(k\varepsilon, t/\varepsilon)_{lj}\, n_j(k, 0) \qquad (7.30)
$$

where

$$
\begin{aligned}
&\tilde{Z}\left(k\varepsilon, \frac{t}{\varepsilon}\right)_{nm} \\
&\quad = \frac{1}{2\pi}\int d\gamma e^{i\gamma t} \left\{\left[\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right)I - C_\varepsilon(k)\right]^{-1} \left\{[I - D_\varepsilon(k)]\right.\right. \\
&\quad \times \left.\left[\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right)I - C_\varepsilon(k)\right]^{-1}\right\}^{-1} \\
&\quad \times D_\varepsilon(\mathbf{k})\left.\left(-\frac{\beta_3}{\varepsilon} + i\gamma\right)I - C_\varepsilon(k)\right]^{-1}\Bigg\}_{nm} \qquad (7.31)
\end{aligned}
$$

The matrix

$$
\{I - D_\varepsilon(k)[(-\beta_3/\varepsilon + i\gamma)I - C_\varepsilon(k)]^{-1}\}^{-1}
$$

is bounded as $-\beta_3/\varepsilon + i\gamma \notin \sigma(A_\varepsilon(k))$; moreover, for large k,

$$
|D_\varepsilon(\mathbf{k})[(-\beta_3/\varepsilon + i\gamma)I - C_\varepsilon(k)]^{-1}]_{nm}| < C
$$

The integral in Eq. (7.31) is then absolutely convergent and defines a bounded matrix operator. It is easy to estimate now

$$\|n_i(\mathbf{r}, t)\|_{H^2(\mathbf{r})} = \left[\int (1+k^2)^2 \, |n_i(\mathbf{k}, t)|^2 \, d_3k \right]^{1/2} < \infty \tag{7.32}$$

This shows that the set $\{n_i(r, t)\}_{i=1}^{5}$ is a solution Eq. (7.7) in $H^2(\mathbf{r})$. ∎

An important property of Eq. (7.7) is given by the following result.

Theorem 7.2. The operator $A_\varepsilon(\nabla)$ for all $\varepsilon \in]0, 1]$ generates a strongly continuous contraction semigroup on $\tilde{H}^2(\mathbf{r}) = \mathsf{X}_{j=1}^{5} H^2(\mathbf{r})$.

Proof. The domain $D(A_\varepsilon(\nabla))$ of the operator $A_\varepsilon(\nabla)$ is dense in $\tilde{H}^2(\mathbf{r})$ and it is enough to show that for $\xi > 0$

$$\| [-A_\varepsilon(\nabla) + \xi I]^{-1} \| \leqslant \xi^{-1} \tag{7.33}$$

then a standard argument (see Kato,[20] p. 479) leads to the conclusion that $A_\varepsilon(\nabla)$ generates a strongly continuous semigroup. For $\phi \in \tilde{H}^2(\mathbf{r})$ we have

$$\begin{aligned} &\{ \| [-A_\varepsilon(\nabla) + \xi I]^{-1} \phi \|_{\tilde{H}^2(\mathbf{r})} \}^2 \\ &\quad = \int d_3k (1+k^2)^2 \sum_{j=1}^{5} |\{ [-A_\varepsilon(\mathbf{k}) + \xi I]^{-1} \phi \}_j |^2 \end{aligned} \tag{7.34}$$

The matrix $(-A_\varepsilon(\mathbf{k}) + \xi I)^{-1}$ is symmetric for all $\mathbf{k} \in \mathbf{R}^3$ and for every $\mathbf{k}$ we can find a unitary matrix $S_\mathbf{k}$ such that

$$S_\mathbf{k}(-A_\varepsilon(\mathbf{k}) + \xi I)^{-1}) S_\mathbf{k}^+ = E(\mathbf{k})$$

and

$$E(k)_{ij} = (-A_i^\varepsilon(k) + \xi)^{-1} \delta_{ij}$$

Denoting $\psi_j = (S_\mathbf{k} \phi)_j$, we can write

$$\sum_{j=1}^{5} |[(-A_\varepsilon(\nabla) + \xi I)^{-1} \phi]_j|^2 = \sum_{j=1}^{5} |[(-A_j^\varepsilon(k) + \xi)^{-1} \psi_j(\mathbf{k})]|^2$$

We can rewrite Eq. (7.33) in the form

$$\begin{aligned} &\int d_3k (1+k^2)^2 \sum_{j=1}^{5} |[(-A_j^\varepsilon(k) + \xi)^{-1} \psi_j(\mathbf{k})]|^2 \\ &\quad \leqslant \int d_3k (1+k^2)^2 \sum_{j=1}^{5} \xi^{-2} |\psi_j(\mathbf{k})|^2 \\ &\quad \leqslant \xi^{-2} (\|\phi\|_{\tilde{H}^2(\mathbf{r})})^2 \end{aligned} \tag{7.35}$$

We have used fact that $A_j^{\varepsilon}(k) \leqslant 0$ for $k \in \mathbf{R}^3$ and the unitary of $S_{\mathbf{k}}$. To see the contraction, observe that for $f \in D(A_\varepsilon(\mathbf{\nabla}))$, we have

$$\frac{d}{dt}\|\exp[A_\varepsilon(\mathbf{\nabla})t]f\|_{\tilde{H}^2(\mathbf{r})}$$

$$=\frac{d}{dt}((1+k^2)\exp[A_\varepsilon^*(\mathbf{k})t]f^*(\mathbf{k}),$$

$$(1+k^2)\exp[A_\varepsilon(\mathbf{k})t]f(k))_{\tilde{L}^2(\mathbf{k})}$$

$$=((1+k^2)\exp[A_\varepsilon^*(\mathbf{k})t]f^*(k),$$

$$2\mathrm{Re}[A_\varepsilon(\mathbf{k}](1+k^2)\exp[A_\varepsilon(\mathbf{k})t]f(\mathbf{k}))_{\tilde{L}^2(\mathbf{k})} \leqslant 0 \qquad (7.36)$$

This last inequality follows from the fact that the operator $\mathrm{Re}[A_\varepsilon(\mathbf{k})]$ is not positive definite, which was proved in proposition 7.1.

The $\tilde{L}^2(\mathbf{k})$ was defined as

$$\tilde{L}^2(\mathbf{k}) = \mathop{\times}_{j=1}^{5} L^2(\mathbf{k}) \qquad (7.37)$$

We have shown that $\exp[A_\varepsilon(\mathbf{\nabla})t]$ is a contraction on $D(A_\varepsilon(\mathbf{\nabla}))$, but as it is a dense set in $\tilde{H}^2(\mathbf{r})$, this extends on the whole space $\tilde{H}^2(\mathbf{r})$. ▮

These two theorems imply the following.

Corollary 7.1. For all $n(\mathbf{r}) \in \tilde{H}^2(\mathbf{r})$, Eq. (7.7) has a unique strong solution $n(\mathbf{r}, t) \in \tilde{H}^2(\mathbf{r})$ and for all $t \geqslant 0$

(i) $$\|n(\mathbf{r}, t)\|_{\tilde{H}^2(\mathbf{r})} \leqslant \|n(\mathbf{r})\|_{\tilde{H}^2(\mathbf{r})} \qquad (7.38)$$

(ii) $$\lim_{t\to\infty} \|n_i(\mathbf{r}, t)\|_{H^2(\mathbf{r})} = 0 \qquad (7.39)$$

Proof. The main assertion of this corollary and (i) follow directly from the fact that $\exp[A_\varepsilon(\nabla)t]$ is a strongly continuous contraction semigroup on $\tilde{H}^2(\mathbf{r})$. Part (ii) can be easily deduced from Theorem 7.1 and the fact that for $k \neq 0$, $\lim_{t\to\infty} n_i(\mathbf{k}, t) = 0$ a.e. ▮

The precise meaning of the hydrodynamic approximation is given by the theorem below. We define an approximation to the solution to the LRBE with initial data $f(\mathbf{r}, \mathbf{p}) \in H_2(\mathbf{r}, \mathbf{p})$ as follows:

$$f_\varepsilon^H(\mathbf{r}, \mathbf{p}, t) = \sum_{j=1}^{5} n_j(\mathbf{r}, t)\, e_{j0}(\mathbf{p}) \tag{7.40}$$

We have then the following result.

Theorem 7.3. Let $f_\varepsilon(\mathbf{r}, \mathbf{p}, t)$ be a solution to the LRBE with initial data $f(\mathbf{r}, \mathbf{p}) \in H_2(\mathbf{r}, \mathbf{p}) \cap L^1(\mathbf{r}, L^2(\mathbf{p}))$ and let $n(\mathbf{r}, t)$ be a solution to Eq. (7.7) with initial data $n_i(\mathbf{r}) = (e_{i0}, f(\mathbf{r}, \mathbf{p}))$; then:

(i)
$$\begin{aligned}&\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - f_\varepsilon^H(\mathbf{r}, \mathbf{p}, t)\|_2 \\ &\quad \leqslant \frac{C}{(1+t)^{5/4}} (\|f(\mathbf{r}, \mathbf{p})\|_2 + \|f(\mathbf{r}, \mathbf{p})\|_{L^1(\mathbf{r}, L^2(\mathbf{p}))})\end{aligned} \tag{7.41}$$

(ii) If in addition we assume that $Qf(\mathbf{r}, \mathbf{p}) = 0$, then

$$\begin{aligned}&\|f(\mathbf{r}, \mathbf{p}, t) - f_\varepsilon^H(\mathbf{r}, \mathbf{p}, t)\|_2 \\ &\quad \leqslant \frac{C}{(1+t)^{7/4}} (\|f(\mathbf{r}, \mathbf{p})\|_2 + \|f(\mathbf{r}, \mathbf{p})\|_{L^1(\mathbf{r}, L^2(\mathbf{p}))})\end{aligned} \tag{7.42}$$

Proof. We choose $\delta = \min(\delta_1, \delta_3)$. Application of Theorems 6.1 and Theorem 7.1 leads to the following explicit expression for $\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - f_\varepsilon^H(\mathbf{r}, \mathbf{p}, t)\|_2$

$$\begin{aligned}
&\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - f_\varepsilon^H(\mathbf{k}, \mathbf{p}, t)\|_2^2 \\
&\quad \leqslant \int d_3 p \left\{ \int_{k\varepsilon < \delta} d_3 k (1+k^2)^2 \left| \sum_{j=1}^{5} \left\{ \exp\left[\frac{t}{\varepsilon} \lambda_j(k\varepsilon) \right] \right. \right. \right. \\
&\quad \times (e_j(-\mathbf{k}\varepsilon), f(\mathbf{k}, \mathbf{p}))\, e_j(\mathbf{k}\varepsilon) - \exp[t\Lambda_j^\varepsilon(k\varepsilon)] \\
&\quad \times \left[(e_{j0}, f(\mathbf{k}, \mathbf{p})) + ik\varepsilon \sum_{l=1}^{5} B_{jl}^1 (e_{l0}, f(\mathbf{k}, \mathbf{p})) \right] e_{j0} \\
&\quad - \sum_{n=1}^{5} \exp[\Lambda_n^\varepsilon(k\varepsilon) t]\, ikB_{jn}^1 (e_{n0}, f(\mathbf{k}, \mathbf{p})) e_{j0} \\
&\quad \left. - \sum_{m=1}^{5} \exp[\Lambda_m^\varepsilon(k\varepsilon) t]\, C_{jm}(k)(e_{m0}, f(\mathbf{k}, \mathbf{p})) e_{j0} \right\} \\
&\quad + \exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon} + \frac{i\mathbf{kp}}{p_0} \right) t \right] f(\mathbf{k}, \mathbf{p}) \\
&\quad \left. + \exp\left(-\frac{\beta_1}{\varepsilon} t \right) Z_1 \left(\mathbf{k}\varepsilon, \frac{t}{\varepsilon} \right) f(\mathbf{k}, \mathbf{p}) \right|^2
\end{aligned}$$

$$+\int_{k\varepsilon>\delta} d_3k(1+k^2)^2 \left|\exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon}+\frac{i\mathbf{kp}}{p_0}\right)t\right]f(\mathbf{k},\mathbf{p})\right.$$

$$+\exp\left(-\frac{\beta_2}{\varepsilon}t\right)Z_2\left(\mathbf{k}\varepsilon,\frac{t}{\varepsilon}\right)f(\mathbf{k},\mathbf{p})$$

$$-\sum_{j=1}^{5}\left\{\exp[-(ik\lambda_{j1}-k^2\varepsilon\lambda_{j2})t]\right.$$

$$\times(e_{j0},f(\mathbf{k},\mathbf{p}))e_{j0}+\exp\left(-\frac{\beta_3}{\varepsilon}t\right)$$

$$\left.\left.\times\sum_{l=1}^{5}\tilde{Z}\left(\mathbf{k}\varepsilon,\frac{t}{\varepsilon}\right)_{jl}(e_{l0},f(\mathbf{k},\mathbf{p}))e_{j0}\right\}\right|^2\Bigg\} \tag{7.43}$$

The second integral can be easily estimated as

$$I_4\leqslant C\exp(-\gamma t)\,\|f(\mathbf{k},\mathbf{p})\|_2^2 \tag{7.44}$$

where $\gamma=\min\{1,\beta_2/\varepsilon,\lambda_{j2}\,\delta^2\varepsilon,\beta_3/\varepsilon\}$.

In order to estimate the first integral in Eq. (7.43), we recall that for $k\varepsilon<\delta$, the explicit forms of $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$, $j=1,\dots,5$, are given by Eqs. (7.2a) and (7.2b), and that of $\Lambda_j^{\varepsilon}(k)$ by Eq. (7.11). Making use of these expressions, we can write the first integral, which denoted I as $I=I_1+I_2+I_3$, where

$$I_1\leqslant C\int d_3p\left\{\int_{k\varepsilon<\delta}d_3k(1+k^2)^2\left|\exp[(ik\lambda_{j1}-k^2\varepsilon\lambda_{j2})t]\right.\right.$$

$$\times\sum_{j=1}^{5}\left[ik\varepsilon(\psi_j,f(\mathbf{k},\mathbf{p}))-\sum_{n=1}^{5}C_{jm}(k)\right.$$

$$\left.\left.\times(e_{m0},f(\mathbf{k},\mathbf{p}))\right]e_{j0}+k^2\varepsilon^2f(\mathbf{k},\mathbf{p})\right|^2\Bigg\} \tag{7.45}$$

$$I_2=\int d_3p\left\{\int_{k\varepsilon<\delta}d_3k(1+k^2)^2\left|\exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon}+\frac{i\mathbf{kp}}{p_0}\right)t\right]f(\mathbf{k},\mathbf{p})\right.\right.$$

$$\left.+\exp\left(-\frac{\beta_1}{\varepsilon}t\right)Z_1\left(\mathbf{k}\varepsilon,\frac{t}{\varepsilon}\right)f(\mathbf{k},\mathbf{p})\right|^2\Bigg\} \tag{7.46}$$

$$I_3 \leqslant C \int d_3 p \left\{ \int_{k\varepsilon < \delta} d_3 k (1+k^2)^2 \left| \exp[(ik\lambda_{j1} - k^2 \varepsilon \lambda_{j2}) \right. \right.$$
$$\times \sum_{j=1}^{5} \left[ik\varepsilon(\psi_j, f(\mathbf{k}, \mathbf{p})) - \sum_{n=1}^{5} C_{jn}(\mathbf{k}) \right.$$
$$\left. \times (e_{n0}, f(\mathbf{k}, \mathbf{p})) e_{n0} \right] + k^2 \varepsilon^2 f(\mathbf{k}, \mathbf{p}) \Big|$$
$$\times \left| \exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon} + \frac{i\mathbf{kp}}{p_0} \right) t \right] f(\mathbf{k}, \mathbf{p}) \right.$$
$$\left. \left. + \exp\left(-\frac{\beta_3}{\varepsilon} t \right) Z_1 \left(\mathbf{k}\varepsilon, \frac{t}{\varepsilon} \right) f(\mathbf{k}, \mathbf{p}) \right| \right\} \tag{7.47}$$

The I_2 and I_3 can be easily estimated and we have

$$I_i \leqslant \exp\left(-\frac{t}{\varepsilon} \right) \| f(\mathbf{k}, \mathbf{p}) \|_2^2, \qquad i = 2, 3 \tag{7.48}$$

The remaining estimate of I_1 can be obtained following the proof of Theorem 6.2(ii), and this results in the decay estimate (i). To prove (ii), we note that the additional assumption $Qf(\mathbf{k}, \mathbf{p}) = 0$ ensures that terms $(\psi_j, f(\mathbf{k}, \mathbf{p}))$ in Eq. (7.45) are equal to zero and the lowest order terms are of order $k^2\varepsilon^2$. This improves our estimate of I_1 and we obtain (ii). ▌

We have shown that $A_\varepsilon(\nabla)$ generates a strongly continuous contraction semigroup on $\tilde{H}^2(\mathbf{r})$, and as it is a dense subspace of $\tilde{L}^2(\mathbf{r}) = \mathsf{X}_{j=1}^5 L^2(\mathbf{r})$, we can define a contraction semigroup $\exp[A_\varepsilon(\nabla)]$ on the whole $\tilde{L}^2(\mathbf{r})$. This means that we now consider a weak solution of Eq. (7.7) and for $f \in L^2(\mathbf{r}, \mathbf{p})$ we define a Navier–Stokes semigroup $N_\varepsilon(t)$ as

$$N_\varepsilon(t) f(\mathbf{r}, \mathbf{p}) = \sum_{j=1}^{5} \left\{ \sum_{l=1}^{5} \exp[A_\varepsilon(\nabla) t]_{jl} (e_{l0}, f(\mathbf{r}, \mathbf{p})) e_{j0} \right\} \tag{7.49}$$

Obviously $N_\varepsilon(t)$ is a strongly continuous contraction semigroup on $L^2(\mathbf{r}, \mathbf{p})$. It is easy to see the following extension of Theorem 7.3.

Theorem 7.4. Let $f_\varepsilon(\mathbf{r}, \mathbf{p}, t)$ be a solution to the LRBE with initial data $f(\mathbf{r}, \mathbf{p}) \in L^2(\mathbf{r}, \mathbf{p}) \cap L^1(\mathbf{r}, L^2(\mathbf{p}))$; then:

(i)
$$\| f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - N_\varepsilon(t) f(\mathbf{r}, \mathbf{p}) \|_{L^2(\mathbf{r}, \mathbf{p})}$$
$$\leqslant \frac{C}{(1+t)^{5/4}} [\| f(\mathbf{r}, \mathbf{p}) \|_{L^2(\mathbf{r}, \mathbf{p})}$$
$$+ \| f(\mathbf{r}, \mathbf{p}) \|_{L^1(\mathbf{r}, L^2(\mathbf{p}))}] \tag{4.50}$$

(ii) If $Qf(\mathbf{r}, \mathbf{p}) = 0$,

$$\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - N_\varepsilon(t) f(\mathbf{r}, \mathbf{p})\|_{L^2(\mathbf{r}, \mathbf{p})} \leqslant \frac{C}{(1+t)^{7/4}} [\|f(\mathbf{r}, \mathbf{p})\|_{L^2(\mathbf{r}, \mathbf{p})} + \|f(\mathbf{r}, \mathbf{p})\|_{L^1(\mathbf{r}, L^2(\mathbf{p}))}] \tag{4.51}$$

Proof. Analogous to the proof of Theorem 7.3.

Note that the decay estimates obtained in Theorem 7.4 are of one order in t better then those obtained for the solution of the LRBE with the same assumptions on the initial data in Theorem 6.3. This shows that indeed the hydrodynamic approximation determines leading terms in the solution to the relativistic Boltzmann equation for long times.

It is worth noting that we have proved the existence of the solution to the hydrodynamic equations (7.7) assuming that the transport coefficients which occur in these equations are derived from the underlying Boltzmann equation. It would of course be interesting to know how much freedom is allowed for these coefficients without these general properties of the solution being lost, but I will not try to answer this question here.

One can try to improve the hydrodynamic approximation by taking into account the higher order terms in perturbation expansions of $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$, but it turns out that the corresponding differential equations are unstable. For example, it is easy to check that coefficients at terms of order $(k\varepsilon)^4$ in the series for $\lambda_j(\mathbf{k}\varepsilon)$ are positive and for large k the solutions of the corresponding differential equations behave like $\exp(\lambda k^4 \varepsilon^3 t)\, n(\mathbf{k}, 0)$ with $\lambda > 0$. The blowup of these solutions shows that the Navier–Stokes equations are very special and in order to improve them we need an additional procedure of including higher order terms to prevent this kind of instability which is connected with higher order terms in the expansion of $\lambda_j(k\varepsilon)$.

Equations of relativistic hydrodynamics have been obtained for the set $\{n_i(\mathbf{r}, t)\}_{i=1}^5$, which are called auxiliary hydrodynamic variables because usually such equations are written for a different set of variables, namely

$$n_0(\mathbf{r}, t) = (f_0^{1/2}, f(\mathbf{r}, \mathbf{p}, t)) \tag{7.52a}$$

$$T^{0k}(\mathbf{r}, t) = (p_k f_0^{1/2}, f(\mathbf{r}, \mathbf{p}, t)) \tag{7.52b}$$

$$T^{00}(\mathbf{r}, t) = (p_0 f_0^{1/2}, f(\mathbf{r}, \mathbf{p}, t)) \tag{7.52c}$$

where the scalar product is again taken in $L^2(\mathbf{p})$. The $n_i(\mathbf{r}, t)$ are linear combinations of these conventional hydrodynamic variables and by suitable transformation of Eq. (7.7) we can obtain corresponding equations for $n_0(\mathbf{r}, t)$, $T^{0k}(\mathbf{r}, t)$, and $T^{00}(\mathbf{r}, t)$. For example, for the rest system of the gas we calculate the explicit form of the transport coefficients in Appendix B and after long but straightforward algebra we obtain

$$\partial_t n_0(\mathbf{r}, t) = -\frac{T^{33}}{S^{033}} \nabla\mathbf{T}(\mathbf{r}, t) + \left[\frac{S^{000}}{NS^{000} - (T^{00})^2} \Gamma_1\right] \Delta n_0(\mathbf{r}, t) - \frac{T^{00}\Gamma_1}{NS^{000} - (T^{00})^2} \Delta T^{00}(\mathbf{r}, t) \tag{7.53a}$$

$$\partial_t T^{00}(\mathbf{r}, t) = -\nabla\mathbf{T}(\mathbf{r}, t) \tag{7.53b}$$

$$\begin{aligned}\nabla_t T(\mathbf{r}, t) = &-\left[\frac{T^{33}}{NS^{000}} - \frac{T^{00}}{N}\left(S^{033} - \frac{T^{00}T^{33}}{N}\right)\right] \nabla n_0(\mathbf{r}, t) \\ &-\left(S^{033} - \frac{T^{00}T^{33}}{N}\right) \nabla T^{00}(\mathbf{r}, t) \\ &+\frac{\Gamma_3}{S^{033}} \Delta T(\mathbf{r}, t) + \frac{\Gamma_2 - \Gamma_3}{S^{033}} \nabla(\nabla\mathbf{T}(\mathbf{r}, t))\end{aligned} \tag{7.53c}$$

where $\mathbf{T}(\mathbf{r}, t)$ denotes a vector with components $T^{0k}(\mathbf{r}, t)$, $k = 1, 2, 3$. The coefficients S^{000}, S^{033}, T^{00}, T^{33}, and N are defined in Appendix B and are connected with thermodynamic properties of the equilibrium state of the gas. An explicit form of the transport coefficients Γ_i is also given in Appendix B.

Note that these equations are in agreement with an exact form of the conservation laws derived from the Boltzmann equation.[10]

8. FLUID DYNAMICAL LIMIT

There is another interesting limit we can consider for the relativistic Boltzmann equation, namely the fluid dynamical limit, i.e., expansion of $f_\varepsilon(\mathbf{r}, \mathbf{p}, t)$ in powers of ε with appropriate rescaling of time variable and then letting $\varepsilon \to 0$. The nonrelativistic results are discussed in ref. 12 and 14. We apply the same strategy to the case of the LRBE.

For the set of hydrodynamic variables $n_i(\mathbf{r}, t)$, $i = 1,..., 5$, we define the Euler equations as

$$\partial_t n_i(\mathbf{r}, t) = E(\nabla)_{ij}\, n_j(\mathbf{r}, t), \qquad i, j = 1,..., 5 \tag{8.1}$$

where

$$E(\nabla)_{nm} = \int d_3k \exp(i\mathbf{kr})\, E(\mathbf{k})_{nm} \tag{8.2}$$

$E(k)$ is a diagonal matrix of the form

$$E(\mathbf{k})_{nm} = ik\lambda_{n1}\,\delta_{nm} \tag{8.3}$$

λ_{n1} are given by Eq. (7.3a).

It is now easy to see the following:

Proposition 8.1. The operator $E(\nabla)$ generates a strongly continuous unitary group $\exp[E(\nabla)t]$ on

$$\tilde{H}^1(\mathbf{r}) = \mathop{\mathsf{X}}_{i=1}^{5} H^1(\mathbf{r}) \tag{8.4}$$

Proof. The explicit form of $\exp[A(\nabla)t]\,\phi(r)$ for $\phi \in \tilde{H}^1(\mathbf{r})$ is

$$\begin{aligned}&\{\exp[E(\nabla)t]\,\phi(\mathbf{r})\}_n\\ &\quad= \int d_3k\,\exp[(ik\lambda_{n1} + i\mathbf{kr})t]\,\phi_n(\mathbf{k})\end{aligned} \tag{8.5}$$

which shows immediately strong continuity and we see that

$$\begin{aligned}\|\exp[E(\nabla)t]\phi\|_{\tilde{H}^1(\mathbf{r})} &= \left[\int d_3k(1+k^2)\sum_{j=1}^{5}|\phi_j(\mathbf{k})|^2\right]^{1/2}\\ &= \|\phi(\mathbf{r})\|_{\tilde{H}^1(\mathbf{r})}\end{aligned} \tag{8.6}$$

These results do not depend on the sign of t. ∎

It follows from Proposition 8.1 that $n(\mathbf{r}, t) \in \tilde{H}^1(\mathbf{r})$ with

$$n_i(\mathbf{r}, t) = \{\exp[E(\nabla)t]\,n(\mathbf{r})\}_i, \qquad i = 1,\ldots, 5 \tag{8.7}$$

is a strong solution to Eq. (8.1) with initial data $n(r) \in \tilde{H}^1(\mathbf{r})$. For $f \in H_1(\mathbf{r}, \mathbf{p})$ we define now an Euler semigroup $E(t)$ as follows:

$$E(t)\,f(\mathbf{r}, \mathbf{p}) = \sum_{n=1}^{5}\sum_{m=1}^{5}[\exp(E(\nabla)t]_{nm}\,(e_{m0}, f(\mathbf{r}, \mathbf{p}))e_{n0} \tag{8.8}$$

This Euler semigroup is a strongly continuous contraction semigroup on $H^1(\mathbf{r}, \mathbf{p})$. This fact is a direct consequence of Proposition 8.1 and definition (8.8). We want to compare the approximate solution $E(t)\,f(\mathbf{r}, \mathbf{p})$ with actual solution of the LRBE with initial data $f(\mathbf{r}, \mathbf{p})$. This is provided by the following theorem.

Theorem 8.1. Let $f_\varepsilon(\mathbf{r}, \mathbf{p}, t)$ be a solution to the LRBE with initial data $f(\mathbf{r}, \mathbf{p}) \in H^3(\mathbf{r}, \mathbf{p})$; then, for $t \geqslant 0$,

$$\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(t)\,f(\mathbf{r}, \mathbf{p})\|_1 \leqslant C\varepsilon\,\|f(\mathbf{r}, \mathbf{p})\|_3 \tag{8.9}$$

Proof. Application of Theorem 6.1 leads to the following explicit expression for $\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(t)\, f(\mathbf{r}, \mathbf{p})\|_1$:

$$\begin{aligned}
&\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(t)\, f(\mathbf{k}, \mathbf{p})\|_1^2 \\
&\leqslant \int d_3 p \left\{ \int_{k\varepsilon < \delta} d_3 k (1 + k^2) \left| \sum_{j=1}^{5} \left[\exp\left[\frac{t}{\varepsilon} \lambda_j(k\varepsilon) \right] \right.\right.\right. \\
&\quad \times (e_j(-k\varepsilon), f(\mathbf{k}, \mathbf{p}))\, e_j(k\varepsilon) \\
&\quad \left. - \exp(t\lambda_{j1})(e_{j0}, f(\mathbf{k}, \mathbf{p}))e_{j0} \right] \\
&\quad + \exp\left[-\left(\frac{\nu(P)}{\varepsilon} + \frac{i\mathbf{kp}}{p_0} \right) t \right] f(\mathbf{k}, \mathbf{p}) \\
&\quad \left. + \exp\left(-\frac{\beta_1}{\varepsilon} t \right) Z_1 \left(\mathbf{k}\varepsilon, \frac{t}{\varepsilon} \right) f(\mathbf{k}, \mathbf{p}) \right|^2 \\
&\quad + \int_{k\varepsilon < \delta} d_3 k (1 + k^2) \left| \exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon} + \frac{i\mathbf{kp}}{p_0} \right) t \right] f(\mathbf{k}, \mathbf{p}) \right. \\
&\quad \times \exp\left(-\frac{\beta_2}{\varepsilon} t \right) Z_2 \left(k\varepsilon, \frac{t}{\varepsilon} \right) f(\mathbf{k}, \mathbf{p}) \\
&\quad \left.\left. - \sum_{j=1}^{5} [\exp[(ik\lambda_{j1})](e_{j0}, f(\mathbf{k}, \mathbf{p}))e_{j0}] \right|^2 \right\}
\end{aligned} \tag{8.10}$$

To estimate the second integral, we use following simple inequality:

$$\begin{aligned}
&\int_{k > k_0} d_3 k (1 + k^2)\, |g(\mathbf{k})|^2 \\
&\qquad \leqslant \int_{k > k_0} d_3 k (1 + k^2) \left(\frac{1 + k^2}{k_0^2} \right)^l |g(\mathbf{k})|^2 \\
&\qquad \leqslant k_0^{-2l} \int_{k > k_0} d_3 k (1 + k^2)^l\, |g(\mathbf{k})|^2 \\
&\qquad \leqslant [k_0^{-l}\, \|g(\mathbf{k})\|_{H^l(\mathbf{k})}]^2
\end{aligned} \tag{8.11}$$

Taking $k_0 = \delta/\varepsilon$, we can denote the second integral in Eq. (8.10) as $I_4 \leqslant \varepsilon^2 C\, \|f(\mathbf{k}, \mathbf{p})\|_2^2$.

In order to estimate the first integral in Eq. (8.10), recall that for $k\varepsilon < \delta$ an explicit form of the $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$, $j = 1, \ldots, 5$, is given by Eqs. (7.2a) and (7.2b). Making use of these expressions, we can write the first integral, denoted I, as $I = I_1 + I_2 + I_3$, where

$$I_1 \leqslant C \int d_3 p \left\{ \int_{k\varepsilon < \delta} d_3 k (1+k^2) \left| \sum_{j=1}^{5} \exp\left[\lambda_j(k\varepsilon) \frac{t}{\varepsilon} \right] \right. \right.$$

$$\times \left[ik\varepsilon(\psi_j, f(\mathbf{k}, \mathbf{p})) + ik\varepsilon \right.$$

$$\left. \times \sum_{l=1}^{5} B_{jl}^1(e_{l0}, f(\mathbf{k}, \mathbf{p})) \right] e_{j0} + k^2 \varepsilon^2 f(\mathbf{k}, \mathbf{p})$$

$$\left. \left. - \sum_{j=1}^{5} \exp[ik\lambda_{j1} t](e_{j0}, f(\mathbf{k}, \mathbf{p})) e_{j0} \right|^2 \right\} \tag{8.12}$$

$$I_2 = \int d_3 p \left\{ \int_{k\varepsilon < \delta} d_3 k (1+k^2) \left| \exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon} + \frac{i\mathbf{kp}}{p_0} \right) t \right] f(\mathbf{k}, \mathbf{p}) \right. \right.$$

$$\left. \left. + \exp\left(-\frac{\beta_1}{\varepsilon} t \right) Z_1 \left(k\varepsilon, \frac{t}{\varepsilon} \right) f(\mathbf{k}, \mathbf{p}) \right|^2 \right\} \tag{8.13}$$

$$I_3 \leqslant C \int d_3 p \left\{ \int_{k\varepsilon < \delta} d_3 k (1+k^2) \left| \sum_{j=1}^{5} \exp\left[\lambda_j(k\varepsilon) \frac{t}{\varepsilon} \right] \right. \right.$$

$$\times \left[ik\varepsilon(\psi_j, f(\mathbf{k}, \mathbf{p})) + ik\varepsilon \right.$$

$$\left. \times \sum_{l=1}^{5} B_{jl}^1(e_{l0}, f(\mathbf{k}, \mathbf{p})) \right] e_{j0} + k^2 \varepsilon^2 f(\mathbf{k}, \mathbf{p})$$

$$\left. - \sum_{j=1}^{5} \exp(ik\lambda_{j1} t)(e_{j0}, f(\mathbf{k}, \mathbf{p})) e_{j0} \right|$$

$$\times \left| \exp\left[-\left(\frac{\nu(\mathbf{p})}{\varepsilon} + \frac{i\mathbf{kp}}{p_0} \right) t \right] f(\mathbf{k}, \mathbf{p}) \right.$$

$$\left. \left. + \exp\left(-\frac{\beta_3}{\varepsilon} t \right) Z_1 \left(k\varepsilon, \frac{t}{\varepsilon} \right) f(\mathbf{k}, \mathbf{p}) \right| \right\} \tag{8.14}$$

The I_2 and I_3 can be easily estimated using Eq. (8.11) and we have, with $\gamma = \min\{1, \beta_1, \beta_3\}$

$$I_i \leqslant C \exp(-\gamma t/\varepsilon) \varepsilon^2 \, \|f(\mathbf{k}, \mathbf{p})\|_2^2, \qquad i = 2, 3 \tag{8.15}$$

Thus, it remains only to estimate I_1. To this end, we observe that for given $t > 0$ and $k^2 \varepsilon t \leqslant \zeta \ll 1$

$$|\exp[\lambda_j(k\varepsilon) t/\varepsilon] - \exp[ik\lambda_{j1} t]| \leqslant C_t \varepsilon (1+k^2) \tag{8.16}$$

We can divide the region of integration in Eq. (8.12) into two parts, the first corresponding to $k^2 < \zeta/\varepsilon t$ and the second to $k\varepsilon < \delta$, $k^2 \geqslant \zeta/\varepsilon t$. In the first region we use Eq. (8.16) and in the second we use the estimate given by Eq. (8.11) with $l = 3$ and choose $k_0 = (\zeta/t\varepsilon)^{1/2}$. These lead to the estimate

$$I_1 \leqslant C_t \varepsilon^2 \, \|f(\mathbf{k}, \mathbf{p})\|_3^2 \tag{8.17}$$

Taking into account that for $l \leqslant m$, $\|f(\mathbf{k}, \mathbf{p})\|_m$, we obtain the desired estimate with constant C_t depending on t. ∎

Remark. The $H^1(\mathbf{r}, \mathbf{p})$ is dense in $L^2(\mathbf{r}, \mathbf{p})$; thus, the notion of the Euler semigroup can be extended to the strongly continuous contraction semigroup on the whole $L^2(\mathbf{r}, \mathbf{p})$ and in this space we can prove an analogous estimate, i.e., for $f(\mathbf{r}, \mathbf{p}) \in H^2(\mathbf{r}, \mathbf{p})$ and $t \geqslant 0$,

$$\|f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(t) f(\mathbf{r}, \mathbf{p})\|_{L^2(\mathbf{r}, \mathbf{p})} \leqslant C\varepsilon \, \|f(\mathbf{r}, \mathbf{p})\|_2 \tag{8.18}$$

The proof is identical to the proof of Theorem 8.1. From Eq. (8.18) we immediately get the following result.

Corollary 8.1. Let $f_\varepsilon(\mathbf{r}, \mathbf{p}, t)$ be a solution to the LRBE with initial data $f(\mathbf{r}, \mathbf{p}) \in L^2(\mathbf{r}, \mathbf{p})$; then, for $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(t) f(\mathbf{r}, \mathbf{p}) = 0 \qquad \text{a.e.} \quad \text{in } L^2(\mathbf{r}, \mathbf{p}) \tag{8.19}$$

This means that the operator $\exp(B_\varepsilon t)$ converges in the limit $\varepsilon = 0$ to the Euler semigroup $E(t)$ in spite of the fact that the damping is absent in the Euler approximation.

The operator $E(\mathbf{V})$ generates a unitary group; thus, the notion of the Euler semigroup can be defined also for $t < 0$. In particular, for $f \in L^2(\mathbf{r}, \mathbf{p})$ the following relation holds ($t \geqslant 0$):

$$E(-t)\, E(t)\, f(\mathbf{r}, \mathbf{p}) = \sum_{j=1}^{5} (e_{j0}, f(\mathbf{r}, \mathbf{p})) e_{j0} \tag{8.20}$$

The next step consists in comparing the Navier–Stokes semigroup $N_\varepsilon(t)$ defined in the previous section and the solution of the Boltzmann equation. As $N_\varepsilon(t)$ depends explicitly on ε, to see its effect in the limit $\varepsilon = 0$ we shall consider simultaneously the long-time limit by rescaling the time variable $t \to t/\varepsilon$. We have the following result.

Theorem 8.2. Let $f_\varepsilon(\mathbf{r}, \mathbf{p}, t)$ be a solution to the LBRE with initial data $f(\mathbf{r}, \mathbf{p}) \in H^2(\mathbf{r}, \mathbf{p})$; then, for $t \geqslant 0$ the following are fulfilled:

(i) $$\|E(-t/\varepsilon) f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(-t/\varepsilon)\, N_\varepsilon(t/\varepsilon)\, f(\mathbf{r}, \mathbf{p})\|_{L^2(\mathbf{r}, \mathbf{p})} \leqslant (\varepsilon \, \|f(\mathbf{r}, \mathbf{p})\|_2 \tag{8.21a}$$

(ii) If we assume that $Qf(\mathbf{r}, \mathbf{p}) = 0$, then

$$\|E(-t/\varepsilon) f_\varepsilon(\mathbf{r}, \mathbf{p}, t) - E(-t/\varepsilon) N_\varepsilon(t/\varepsilon) f(\mathbf{r}, \mathbf{p})\|_{L^2(\mathbf{r}, \mathbf{p})} \leqslant C\varepsilon^2 \|f(\mathbf{r}, \mathbf{p})\|_2 \tag{8.21b}$$

Proof. The proof is similar to the proof of Theorem 8.1.

9. SUMMARY AND CONCLUSIONS

I have shown that for hard interactions for sufficiently long times and/or small mean free paths the solution to the linearized relativistic Boltzmann equation can be approximated by the solution of the set of five differential equations for conserved variables only. This left open one important problem. As these equations are of parabolic type, there are well-known difficulties with an infinite speed of propagation and the Lorentz transformation.[3-5] The present results state only that if any observer moving relative to the gas formulates a Cauchy problem for the Boltzmann equation in his frame, he can find a set of differential equations with corresponding initial data and then form with the help of solutions to these equations an approximation to the solution to the Boltzmann equation valid for sufficiently long times as measured in his frame. On the other hand, if we are interested in comparing approximations made by different observers, we need of course to know how these equations of hydrodynamics transform from one frame to the other. But if one specifies in one frame the initial data on the surfaces $t = 0$, then when seen from the other, moving frame, these data are no longer of the Cauchy type, but rather are specified on some spacelike surface. Thus, in order to solve this problem, one should find first a solution to the LRBE with such initial data and next check if there exists a corresponding hydrodynamic approximation, which means performing the same analysis we made for the Cauchy problem.

It should be stressed that the present results are not conclusive in the sense that we were able only to analyse the explicit form of the solution in the long-time limit or zero-mean-free-path limit. In these limits one is left with the leading terms only, which results in a breakdown of the causal structure of the relativistic Boltzmann equation. I have recently discussed this problem for a model Boltzmann equation,[23] where I have shown that teven a hyperbolic approximation to this equation converges in both limits to the corresponding parabolic equation of hydrodynamics. On the other hand, it is clear from the results obtained so far that apart from the five conserved variables, no other can be sorted out of the whole set of nonhydrodynamic variables as is done in the derivation of extended hydrodynamics or in the Grad methods of moments.[2,4,5] In this sense the present results are close to the approach of van Kampen.[6]

APPENDIX A

I present here the details of the perturbation expansion for the eigenvalues $\lambda_i(k\varepsilon)$ and eigenfunctions $e_i(\mathbf{k}\varepsilon)$ of the operator $B_{k\varepsilon}$ for $i=1,..., 5$.

The set $N_0 = \ker(L)$ is spanned by five independent but in general not orthogonal functions $f_0^{1/2}$, $p_x f_0^{1/2}$, $p_y f_0^{1/2}$, $p_z f_0^{1/2}$, and $p_0 f_0^{1/2}$. Denote by $\{\phi_i\}_{i=1}^5$ an orthonormal set of functions obtained from the above set with a Gram–Schmidt procedure. Now consider the eigenvalue problem for the operator $B_{k\varepsilon}$

$$ik\varepsilon E\psi_j(\mathbf{k}\varepsilon) + L\psi_j(\mathbf{k}\varepsilon) = \lambda_j(k\varepsilon)\,\psi_j(k\varepsilon) \tag{A.1}$$

where

$$E = -\mathbf{kp}/kp_0 \tag{A.2}$$

We treat $k\varepsilon$ as a small parameter. Taking a scalar product in $L^2(r)$ of Eq. (A.1) with $\psi_j(k\varepsilon)$ and using the fact that the operator L is independent of $\mathbf{k}$, we see that $\lambda_j(k\varepsilon)$ is indeed a function of $k\varepsilon$ alone, which we have already anticipated in our notation. We expand now the eigenvalues and eigenfunctions of $B_{k\varepsilon}$ as follows:

$$\lambda_j(k\varepsilon) = \sum_{n=0}^{\infty} \lambda_{jn}(ik\varepsilon)^n \tag{A.3}$$

$$e_j(\mathbf{k}\varepsilon) = \sum_{n=0}^{\infty} e_{jn}(ik\varepsilon)^n \tag{A.4}$$

The existence of such expansions is assured by the Kato–Rellich theorem. We are interested in calculating coefficients λ_{jn} and e_{jn} for $j=1,..., 5$. Making use of these representations of $\lambda_j(k\varepsilon)$ and $e_j(\mathbf{k}\varepsilon)$ in Eq. (A.1) and comparing terms of the same order in $k\varepsilon$, we obtain

$$Le_{j0} = \lambda_{j0} e_{j0} \tag{A.5}$$

$$Le_{jn} = \sum_{m=0}^{5} \lambda_{jn} e_{jn-m} - Ee_{jn-1}, \qquad n \leqslant 1 \tag{A.6}$$

For $j=1,..., 5$, $Le_{j0}=0$. As the zereo eigenvalue of L is fivefold degenerate, we assume that the e_{j0} are linear combinations of $\{\phi_l\}_{l=1}^5$ in the form

$$e_{j0} = \sum_{l=1}^{5} A_{jl}\phi_l, \qquad j=1,..., 5 \tag{A.7}$$

The coefficients A_{jl} are chosen in such a way that the operator E is diagonal in e_{j0},

$$(e_{j0}, Ee_{l0}) = \lambda_{j1}\, \delta_{jl}, \qquad j, l = 1,..., 5 \tag{A.8}$$

These λ_{j1} and A_{jl} can be calculated as the eigenvalues and corresponding eigenvectors of the following matrix equation:

$$(\lambda I - M)\, \bar{A}^i = 0 \tag{A.9}$$

where

$$(\bar{A}^i)^T = [A_{i1},..., A_{i5}] \tag{A.10a}$$

$$A_{ij} = (\phi_i, E\phi_j) \tag{A.10b}$$

$M_{ij} \in R$, $M_{ij} = M_{ji}$ for $i \neq j$ and it is easy to see that the matrix M is diagonalizable and that λ_{i1} and the corresponding $\bar{A}^i$ are real. As the $\|E\| < 1$, we see also that $|\lambda_{i1}| < 1$.

Note that in the case when the degeneracy is not removed by the operator E in the first-order perturbation, i.e., $\lambda_{i1} = \lambda_{j1}$ for some pair $i, j \in (1,..., 5)$, we can always choose these e_{i0} and e_{j0} such that they also diagonalize the operator $E(QLQ)^{-1} E$, where $Q = 1 - P$ and P is a projection on N_0.

The first-order corrections to e_{j0} can be calculated from the following equation:

$$L\psi_{j1} = (\lambda_{j1} - E)\psi_{j0} \tag{A.11}$$

The solution of this equation has the form

$$\psi_{j1} = -(QLQ)^{-1}\, E\psi_{j0} \tag{A.12}$$

We can add to this solution a combination of the solutions of the homogeneous equation $L\psi = 0$. We write then the general form of e_{j1} as

$$e_{j1} = -(QLQ)^{-1}\, Ee_{j0} + \sum_{l=1}^{5} B^1_{jl} e_{l0} \tag{A.13}$$

with $B^1_{jj} = 0$.

Coefficients B^1_{kl} are fixed by the requirement that the rhs of the equation for the second-order corrections is perpendicular to N_0, i.e.,

$$L\psi_{j2} = (\lambda_{j1} - E)e_{j1} + \lambda_{j2} e_{j2} \tag{A.14}$$

Taking the scalar product of this equation with e_{l0}, $l=1,\dots, 5$, we obtain

$$\lambda_{j2} = -(e_{j0}, E(QLQ)^{-1} Ee_{j0}) \tag{A.15}$$

$$B^1_{jl} = \begin{cases} -(\lambda_{j1} - \lambda_{l1})^{-1} (e_{l0}, E(QLQ)^{-1} Ee_{j0}), & i \neq j, \lambda_{j1} \neq \lambda_{l1} \\ 0, & \text{otherwise} \end{cases} \tag{A.16}$$

The operator $(QLQ)^{-1}$ is not positive definite; thus, the $\lambda_{j2} \geqslant 0$. In fact, we need to show that $\lambda_{j2} > 0$ for $j=1,\dots, 5$. To see this, note that $(QLQ)^{-1} E\psi = 0$ iff $\psi = \alpha p_0 f_0^{1/2}$. We assume then that one of e_{j0}, say $e_{10} = \alpha p_0 f_0^{1/2}$. Without loss of generality, we can take the z axis parallel to the vector k and we see then that $Ee_{10} = \alpha p_z f_0^{1/2}$. Taking now the scalar product of Ee_{l0} with e_{j0}, $j=2,\dots, 5$, we see that all these e_{j0} are orthogonal to $p_z f_0^{1/2}$ and, as they together with e_{10} form an orthonormal set, also to $p_0 f_0^{1/2}$. It is easy to check that the subspace orthogonal to $\{p_0 f_0^{1/2}, p_z f_0^{1/2}\}$ in N_0 is only three dimensional; thus, the set $\{e_{j0}\}_{j=2}^{5}$ cannot be linearly independent, which contradicts the fact that they form an orthonormal basis. We see then that the e_{j0} must be either orthogonal to $p_0 f_0^{1/2}$ or be of the form $e_{j0} = \alpha p_0 f_0^{1/2} + \beta\psi$ with $(\psi, p_0 f_0^{1/2}) = 0$. Thus, $-\lambda_{j2} \leqslant \sup_j (e_{j0} E, (QLQ)^{-1} Ee_{j0}) < 0$.

It follows from the remark above that for $\lambda_{i0} = \lambda_{j0}$, $i \neq j$, we have

$$A_{ij} = (e_{j0}, E(QLQ)^{-1} Ee_{i0}) \equiv 0 \tag{A.17}$$

In general we obtain for $n \geqslant 1$

$$e_{j0} = -(QLQ)^{-1} Ee_{jn-1} + \sum_{l=1}^{5} B^n_{jl} e_{l0} \tag{A.18}$$

with $B^n_{jj} = 0$.

For B^n_{ij} and λ_{in} we obtain two coupled recursive relations:

$$\lambda_{in} = (e_{i0}, E(-(QLQ)^{-1} E)^{n-1} e_{i0}) + \sum_{m=1}^{n-2} \sum_{j=1}^{5} B^{n-m-1}_{ij} (e_{i0}, E(-(QLQ)^{-1} E)^m e_{j0}) \tag{A.19}$$

$$\begin{aligned}(\lambda_{k1} - \lambda_{i1}) B^n_{ik} = {} & \sum_{m=2}^{n+1} \lambda_{im} B^{n+1-m}_{ik} \\ & - \sum_{p=1}^{n-1} \sum_{j=1}^{5} B^{n-p}_{ij} (e_{k0}, E(-(QLQ)^{-1} E)^p e_{j0}) \\ & - (e_{k0}, E(-(QLQ)^{-1} E)^n e_{i0})\end{aligned} \tag{A.20}$$

The vectors $e_j(\mathbf{k}\varepsilon) = \sum_{n=0}^{\infty} e_{jn}(ik\varepsilon)^n$ have to be normalized. Let us choose a normalization such that

$$(e_i(-k\varepsilon), e_j(k\varepsilon)) = \delta_{ij} \tag{A.21}$$

For $k\varepsilon$ small enough, we can write

$$\lambda_j(k\varepsilon) = ik\varepsilon\lambda_{j1} - k^2\varepsilon^2\lambda_{j2} + o(k^2\varepsilon^2)$$

$$e_j(\mathbf{k}\varepsilon) = e_{j0} + ik\varepsilon \sum_{l=1}^{5} B^1_{jl} e_{l0} + ik\varepsilon\psi_{j1} + o(k\varepsilon)$$

where

$$\psi_{j1} = -(QLQ)^{-1} Ee_{j0}$$

We see that $(\psi_{j1}, e_{l0}) \equiv 0$, j, $l = 1,..., 5$.

APPENDIX B

Here I calculate the explicit form of the coefficients λ_{in} and e_{jn} in the rest frame of the gas. In such a frame the hydrodynamic four-velocity u^μ has the simple form $u^\mu = [1, 0, 0, 0]$ and $f_0 = C\exp(-p_0)$. The orthonormal set $\{\phi_i\}$ which spans N_0 can be chosen as

$$\phi_{10} = f_0^{1/2} N^{-1/2} \tag{B.1a}$$

$$\phi_{20} = p_x f_0^{1/2} (S^{001})^{-1/2} \tag{B.1b}$$

$$\phi_{30} = p_y f_0^{1/2} (S^{002})^{-1/2} \tag{B.1c}$$

$$\phi_{40} = p_z f_0^{1/2} (S^{003})^{-1/2} \tag{B.1d}$$

$$\phi_{50} = \left(p_0 f_0^{1/2} - \frac{T^{00}}{N} f_0^{1/2} \right) \left(S^{000} - \frac{(T^{00})^2}{N} \right)^{-1/2} \tag{B.1e}$$

where

$$N = \int d_3 p f_0 \tag{B.2a}$$

$$T^{\alpha\beta} = \int d_3 p \frac{p_\alpha p_\beta}{p_0} f_0 \tag{B.2b}$$

$$S^{0\alpha\beta} = \int d_3 p p_\alpha p_\beta f_0 \qquad (\alpha, \beta = 0, 1, 2, 3) \tag{B.2c}$$

From the symmetry of f_0 with respect to interchange p_i $i=1, 2, 3$, we see that $S^{011}=S^{022}=S^{033}$. The matrix equation determining both λ_{i1} and $\bar{A}^i$ has the form

$$(M-\lambda I)\bar{A}^i=0 \tag{B.3}$$

where $M_{ij}=M_{ji}$, and for $i>j$,

$$M_{pr}=\begin{cases}(\phi_{p0}, (p_z/p_0)\phi_{q0}), & p=1, q=4 \quad \text{or} \quad p=4, q=5\\ 0, & \text{otherwise}\end{cases} \tag{B.4}$$

The solution of this equation leads to the following eigenvalues and eigenvectors:

$$\lambda_{11} = -[(\beta_{14})^2+(\beta_{45})^2]^{1/2}=-\lambda \tag{B.5a}$$

$$e_{10}=2^{-1/2}\left(\frac{\beta_{14}}{\lambda}\phi_{10}+\phi_{40}+\frac{\beta_{45}}{\lambda}\phi_{50}\right) \tag{B.5b}$$

$$\lambda_{21}=\lambda \tag{B.6a}$$

$$e_{20}=2^{-1/2}\left(-\frac{\beta_{14}}{\lambda}\phi_{10}+\phi_{40}-\frac{\beta_{45}}{\lambda}\phi_{50}\right) \tag{B.6b}$$

$$\lambda_{31}=0 \tag{B.7a}$$

$$e_{30}=\frac{\beta_{45}}{\lambda}\phi_{10}-\frac{\beta_{14}}{\lambda}\phi_{50} \tag{B.7b}$$

$$\lambda_{41}=0 \tag{B.8a}$$

$$e_{40}=\phi_{20} \tag{B.8b}$$

$$\lambda_{51}=0 \tag{B.9a}$$

$$e_{50}=\phi_{30} \tag{B.9b}$$

where

$$\beta_{14}=\int d_3 p p_z^2 p_0^{-1} f_0 (NS^{033})^{-1/2}=T^{33}(NS^{033})^{-1/2} \tag{B.10}$$

$$\beta_{45}=\beta_{45}^1-\beta_{45}^2 \tag{B.11}$$

$$\beta_{45}^1=\int d_3 p p_z^2 f_0\left[S^{033}-\left(S^{000}-\frac{(T^{00})^2}{N}\right)\right]^{-1/2}$$
$$=\left[\frac{S^{033}N}{NS^{000}-(T^{00})^2}\right]^{1/2} \tag{B.12}$$

$$\beta_{45}^2 = \int d_3 p \frac{p_z^2}{p_0} f_0 \left[\left(S^{000} - \frac{(T^{00})^2}{N} \right) S^{033} \right]^{-1/2}$$

$$= \frac{T^{00} T^{33}}{(NS^{033})^{1/2}} [NS^{000} - (T^{00})^2]^{-1/2} \tag{B.13}$$

In the next order in $k\varepsilon$ we obtain

$$\lambda_{12} = \frac{1}{2} \left[\left\{ \frac{\beta_{14}}{\lambda N^{1/2}} - \frac{\beta_{45} T^{00}}{\lambda N^{1/2}} [NS^{000} - (T^{00})^2]^{-1/2} \right\}^2 \Gamma_1 \right.$$

$$\left. + (S^{033})^{-1} \Gamma_2 \right] \tag{B.14a}$$

$$B_{12}^1 = \frac{1}{2\lambda} \left[\left\{ \frac{\beta_{14}}{\lambda} N^{-1/2} - \frac{\beta_{45}}{\lambda} T^{00} N^{-1/2} \right. \right.$$

$$\left. \left. \times [NS^{000} - (T^{00})^2]^{-1/2} \right\}^2 \Gamma_1 + (S^{033})^{-1} \Gamma_2 \right] \tag{B.14b}$$

$$B_{13}^1 = \frac{1}{2^{1/2}\lambda} \left[\left\{ \frac{\beta_{14}}{\lambda} N^{-1/2} - \frac{\beta_{45}}{\lambda} T^{00} N^{-1/2} \right. \right.$$

$$\left. \times [NS^{000} - (T^{00})^2]^{-1/2} \right\} \times \left\{ \frac{\beta_{45}}{\lambda} N^{-1/2} \right.$$

$$\left. \left. + \frac{\beta_{14}}{\lambda} T^{00} N^{-1/2} [NS^{000} - (T^{00})^2]^{-1/2} \right\} \Gamma_1 \right] \tag{B.14c}$$

$$e_{11} = -(QLQ)^{-1} \frac{p_z}{p_0} e_{10} + B_{12}^1 e_{20} + B_{13}^1 e_{30} \tag{B.14d}$$

$$\lambda_{22} = \frac{1}{2} \left[\left\{ \frac{\beta_{14}}{\lambda} N^{-1/2} - \frac{\beta_{45}}{\lambda} T^{00} N^{-1/2} [NS^{000} - (T^{00})^2]^{-1/2} \right\} \Gamma_1 \right.$$

$$\left. + (S^{033})^{-1} \Gamma_2 \right] \tag{B.15a}$$

$$B_{23}^1 = -B_{13}^1 \tag{B.15b}$$

$$e_{21} = -(QLQ)^{-1} \frac{p_z}{p_0} e_{20} + B_{12}^1 e_{10} + B_{23}^1 e_{30} \tag{B.15c}$$

$$\lambda_{32} = \left\{ \frac{\beta_{45}}{\lambda} N^{-1/2} + \frac{\beta_{14}}{\lambda} T^{00} N^{-1/2} \right.$$

$$\left. \times [NS^{000} - (T^{00})^2]^{-1/2} \right\} \Gamma_1 \tag{B.16a}$$

$$e_{31} = -(QLQ)^{-1} \frac{p_z}{p_0} e_{30} + B^1_{13} e_{30} + B^1_{23} e_{20} \tag{B.16b}$$

$$\lambda_{42} = (S^{011})^{-1} \Gamma_3 \tag{B.17a}$$

$$e_{41} = -(QLQ)^{-1} \frac{p_z}{p_0} e_{40} \tag{B.17b}$$

$$\lambda_{52} = (S^{022})^{-1} \Gamma_3 \tag{B.18a}$$

$$e_{51} = -(QLQ)^{-1} \frac{p_z}{p_0} e_{50} \tag{B.18b}$$

where

$$\Gamma_1 = -\left(f_0^{1/2} \frac{p_z}{p_0}, (QLQ)^{-1} \frac{p_z}{p_0} f_0^{1/2}\right) \tag{B.19a}$$

$$\Gamma_2 = -\left(f_0^{1/2} \frac{p_z^2}{p_0}, (QLQ)^{-1} \frac{p_z^2}{p_0} f_0^{1/2}\right) \tag{B.19b}$$

$$\Gamma_3 = -\left(f_0^{1/2} \frac{p_z p_x}{p_0}, (QLQ)^{-1} \frac{p_z p_x}{p_0} f_0^{1/2}\right) \tag{B.19c}$$

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